Long-time dynamical behaviour of weakly interacting lattice fermions

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Outline

Weakly interacting lattice fermions

We consider a system of fermions on the lattice $\Omega = \{-L/2, \ldots, 0, \ldots, L/2\}^d$ ($d \geq 1$) interacting via a two-body potential $\lambda V(x-y)$. The function V is fixed and the small constant $\lambda \to 0^+$ is used to model the weakly interacting regime. We work in the grand-canonical ensemble on the fermionic Fock space

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The set of functions $\{e_x = \mathbb{1}_{\{x\}} \mid x \in \Omega\}$ is an orthonormal basis of $\ell^2(\Omega)$. Recall the annihilation and creation operators $a(x)$, $a(x)^*$ defined by

$$a(x)(\psi_1 \otimes \cdots \otimes \psi_n) = \langle e_x, \psi_1 \rangle \psi_2 \otimes \cdots \otimes \psi_n,$$

$$a(x)^*(\psi_1 \otimes \cdots \otimes \psi_n) = e_x \otimes \psi_1 \otimes \cdots \otimes \psi_n.$$

They extend naturally to all of $\mathcal{F}$ with $\|a(x)\| = \|a(x)^*\| = 1$, and they satisfy the anti-commutation relations

$$\{a(x), a(y)^*\} = \delta(x-y), \quad \{a(x), a(y)\} = 0, \quad \{a(x)^*, a(y)^*\} = 0$$

where $\{A, B\} = AB + BA$.

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where $\{A, B\} = AB + BA$. The Hamiltonian of our system then reads

$$H = \sum_{x,y\in\Omega} \alpha(x-y) a(x)^* a(y) + \lambda \sum_{x,y\in\Omega} V(x-y) a(x)^* a(y)^* a(y) a(x),$$

where $\alpha : \mathbb{Z}^d \to \mathbb{R}$ is the hopping amplitude.

It is convenient to introduce the Fourier transform

$$\hat{f}(k) = \sum_{x \in \Omega} f(x) e^{-2\pi i k \cdot x}, \quad k \in \Omega^* = \left\{ -\frac{L-1}{2L}, \ldots, 0, \ldots, \frac{L-1}{2L} \right\}^d,$$

and instead work in momentum space with $\hat{a}(k)$ and $\hat{a}(k)^*$, $k \in \Omega^*$. The dual lattice Ω^* is a subset of the d-dimensional torus $\mathbb{T}^d = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ and formally $\Omega^* \to \mathbb{T}^d$ as $L \to \infty$. We will use the shorthand notation

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The Hamiltonian can be rewritten as

$$H = \int_{\Omega^*} \mathrm{d}k \omega(k) \hat{a}(k)^* \hat{a}(k) + \lambda \iiiint_{(\Omega^*)^4} \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 + k_2 - k_3 - k_4) \times \\ \times \hat{V}(k_2 - k_3) \hat{a}(k_1)^* \hat{a}(k_2)^* \hat{a}(k_3) \hat{a}(k_4),$$

where $\omega(k) = \hat{\alpha}(k)$ is the *dispersion relation*.

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where $\omega(k) = \hat{\alpha}(k)$ is the *dispersion relation*. Think of the simple case of nearest neighbour hopping, where

$$\omega(k) = c - \sum_{j=1}^{d} \cos(2\pi k_j),$$

where the constant c is arbitrary, and k_j denotes the j'th coordinate of $k \in \mathbb{T}^d$.

Two-point time correlation function

For any operator A on Fock space, its Heisenberg evolution is defined by

$$A(t) = e^{itH} A e^{-itH}.$$

This satisfies the equation

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In particular, for the annihilation operators we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{a}(k_1, t) = -i\omega(k_1)\hat{a}(k_1, t) - i\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 + k_2 - k_3 - k_4) \times \\ \times \hat{V}(k_2 - k_3) \hat{a}(k_2, t)^* \hat{a}(k_3, t) \hat{a}(k_4, t). \tag{1}$$

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Given an initial state $\langle \cdot \rangle$, we are interested in the two-point time correlation function $\langle \hat{a}(k)^* \hat{a}(k', t) \rangle$. We will restrict the attention to the initial Gibbs state $Z^{-1} e^{-H}$, namely

$$\langle A \rangle = \frac{1}{Z} \mathrm{Tr}(e^{-H} A), \quad Z = \mathrm{Tr} e^{-H}.$$

- When the initial state $\langle \cdot \rangle$ is *translation* and *gauge* invariant ($\langle e^{i\theta N} A e^{-i\theta N} \rangle = \langle A \rangle$), the two-point correlation function is of the form

$$\langle a(k,t)^* a(k',t) \rangle = \delta(k-k') W_\lambda(k,t)$$

for some $0 \leq W_\lambda(\,\cdot\,, t) \leq 1$ defined on Ω^*.

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- As mentioned, we will restrict ourselves to thermal equlibrium, meaning that W_λ is also *time independent*,

$$\langle a(k,t)^* a(k',t) \rangle = \langle a(k)^* a(k') \rangle = \delta(k-k') W_\lambda(k).$$

- If $\lambda = 0$ (the non-interacting gas) and $t = 0$, the two-point correlation function is simply

$$\langle \hat{a}(k)^* \hat{a}(k') \rangle_0 = \delta(k-k') W_\infty(k), \quad W_\infty(k) = \frac{1}{e^{\omega(k)}+1}.$$

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- Rigorously deriving effective descriptions of $W_\lambda(k,t)$ is a long-standing open problem. Our goal is to prove an effective description for the two-point time correlation function $\langle \hat{a}(k)^* \hat{a}(k',t) \rangle$ in the weakly interacting regime $\lambda \to 0^+$ up to the kinetic time $t \sim \lambda^{-2}$.

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Literature:

- Theoretical physics: Nordheim '28, Peierls '29, Van Hove '54, Prigogine '62, Balescu '75, Hugenholtz '83...
- Using Duhamel expansions: Peierls '29, Lanford '75, Davies '76, Spohn '77, Erdös-Salmhofer-Yau '08, Lukkarinen-Spohn '09...
- Recent development by Deng-Hani in kinetic wave theory ('22+).

Boltzmann-Nordheim kinetic equation

Kinetic conjecture: Under suitable assumptions on ω and the state $\langle \cdot \rangle$,

$$\boxed{\lim_{\lambda \to 0} \lim_{L \to \infty} W_\lambda(k, \lambda^{-2} t) = W(k,t),}$$

where $W(k,t)$ solves the homogeneous Boltzmann-Nordheim equation

$$\frac{\partial}{\partial t} W(k,t) = \mathcal{C}\big(W(\,\cdot\,, t)\big)(k), \tag{2}$$

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$$\frac{\partial}{\partial t}W(k,t) = \mathcal{C}\big(W(\,\cdot\,,t)\big)(k), \tag{2}$$

with the collision operator

$$\mathcal{C}(W)(k_1) = \int_{\mathbb{T}^{3d}} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\delta(k_1+k_2-k_3-k_4)\delta(\omega(k_1)+\omega(k_2)-\omega(k_3)-\omega(k_4))$$
$$\times\,|\widehat{V}(k_2-k_3)-\widehat{V}(k_2-k_4)|^2\big((1-W_1)(1-W_2)W_3W_4 - W_1W_2(1-W_3)(1-W_4)\big).$$

Here we have introduced the shorthand notation $W_j = W(k_j)$, $j = 1,2,3,4$. W_∞ is the equilibrium solution to (2),

$$\mathcal{C}(W_\infty) = 0,$$

so one could expect $W(\,\cdot\,,t)\to W_\infty$ as $t\to\infty$.

For any compactly supported $f, g \in \ell^2(\mathbb{Z}^d)$, we denote by $\hat{f}$, $\hat{g}$ their Fourier transforms and consider the following testing two-point correlation operator

$$Q^\lambda[g,f](t) = \iint_{(\Omega^*)^2} \mathrm{d}k\mathrm{d}k' \hat{g}(k)^* \hat{f}(k') \langle \hat{a}(k',0)^* e^{i\omega^\lambda(k)t/\lambda^2} \hat{a}(k, t/\lambda^2) \rangle$$

where $\langle \cdot \rangle$ is the Gibbs state $Z^{-1}e^{-H}$ and $\omega_\lambda(k)$ is the modified dispersion relation

$$\omega^\lambda = \omega + \lambda R_\lambda,$$

with

$$R_\lambda = \int_{\Omega^*} \mathrm{d}k_2 W_\lambda(k_2)[\hat{V}(0) - \hat{V}(k_1 - k_2)].$$

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Under additional technical assumptions on the decay of equilibrium correlations and and on the dispersion relation ω, we expect to be able to prove

Theorem (Conjecture)

Suppose that $d \geq 4$. There exists $t_0 > 0$ such that for all $|t| < t_0$ we have

$$\lim_{\lambda \to 0} \limsup_{L \to \infty} \left| Q^\lambda[g,f](t) - \int_{\mathbb{T}^d} \mathrm{d}k \hat{g}(k)^* \hat{f}(k) W_\infty(k) e^{-\nu_1(k)|t| - it\nu_2(k)} \right| = 0,$$

where ν_1, ν_2 are real functions defined by a quantity which is proportional to the collisional frequency μ_∞ of the Boltzmann-Nordheim operator at equilibrium (they are related to W_∞).

- The modified dispersion relation ω^λ is introduced to handle effects of order λ at times $\lambda^{-1} \lesssim t \lesssim \lambda^{-2}$
- Loosely speaking, the "theorem" says that for not too large $t \sim \mathcal{O}(\lambda^{-2})$,

$$\langle \hat{a}(k',0)^* \hat{a}(k,t) \rangle \approx \delta(k-k') W_\infty(k) e^{-it(w^\lambda(k)+\lambda^2 \nu_2(k))} e^{-\lambda^2 t \nu_1(k)}.$$

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- Structurally, our problem ressembles a similar problem for the non-linear Schrödinger equation (formally replace $\hat{a}(k,t)$ by a *classical* field $\hat{\psi}_t(k)$). Our strategy is closely inspired by (Lukkarien-Spohn '09), where the NLS is treated in the weak coupling regime.
- The main strategy is to formally expand $Q^\lambda[g,f](t)$ in a series (Duhamel expansion)

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- The terms $\mathcal{Z}_n(t)$ can be represented using Feynman diagrams. We re-use the careful classification of the appearing diagrams that was done in (Lukkarinen-Spohn '09).
- One needs to stop the expansion at a suitable N (probably $N \sim -\log(\lambda)$), and then prove term by term convergence.

Duhamel expansion

We return to the Duhamel formula (1). Introduce cut-off functions $\Phi_1, \Phi_2 \in C^\infty((\mathbb{T}^d)^3)$ with $1 = \Phi_0 + \Phi_1$, and further satisfying that if $k_i + k_j = 0$, then $\Psi_1 = 0$ and $\Phi_0 = 1$.

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{a}(k_1,t) = & -i\omega(k_1)\hat{a}(k_1,t) - i\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\delta(k_1-k_2-k_3-k_4) \\
& \qquad \times \hat{V}(k_2+k_3)\Phi_0(k_2,k_3,k_4)\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t) \\
& - i\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\delta(k_1-k_2-k_3-k_4) \\
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$$\begin{aligned}\frac{\partial}{\partial t}\hat{a}(k_1,t) = & -i\omega(k_1)\hat{a}(k_1,t) - i\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\delta(k_1-k_2-k_3-k_4) \\ & \qquad \times \hat{V}(k_2+k_3)\Phi_0(k_2,k_3,k_4)\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t) \\ & - i\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4\delta(k_1-k_2-k_3-k_4) \\ & \qquad \times \hat{V}(k_2+k_3)\Phi_1(k_2,k_3,k_4)\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t).\end{aligned}$$

Let us now write

$$\begin{aligned}\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t) &= \hat{a}(k_4,t)\langle\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\rangle - \hat{a}(k_3,t)\langle\hat{a}(-k_2,t)^*\hat{a}(k_4,t)\rangle \\ &\quad + [\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t)]^T \\ &= \hat{a}(k_4,t)\delta(k_2+k_3)W_\lambda(-k_2) - \hat{a}(k_3,t)\delta(k_2+k_4)W_\lambda(-k_2) \\ &\quad + [\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t)]^T\end{aligned}$$

and insert into the term containing Φ_0.

$$\frac{\partial}{\partial t}\hat{a}(k_1,t) = -i\hat{a}(k_1,t)\left(\omega(k_1) + \lambda\int_{\Omega^*} \mathrm{d}k_2 W_\lambda(k_2)[\hat{V}(0) - \hat{V}(k_1-k_2)]\right)$$
$$- i\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4 \delta(k_1-k_2-k_3-k_4)\hat{V}(k_2+k_3)$$
$$\times [\Phi_1(k_2,k_3,k_4)a(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t) + \Phi_0(k_2,k_3,k_4)[\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t)]^T]. \quad (3)$$

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$$- i\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4 \delta(k_1-k_2-k_3-k_4)\hat{V}(k_2+k_3)$$
$$\times [\Phi_1(k_2,k_3,k_4)a(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t) + \Phi_0(k_2,k_3,k_4)[\hat{a}(-k_2,t)^*\hat{a}(k_3,t)\hat{a}(k_4,t)]^T]. \quad (3)$$

We now define

$$R_\lambda(k_1) = \int_{\Omega^*} \mathrm{d}k_2 W_\lambda(k_2)[\hat{V}(0) - \hat{V}(k_1-k_2)],$$

and the modified dispersion relation

$$\omega^\lambda = \omega + \lambda R_\lambda,$$

whose role is to absorb the terms containing $\hat{a}(k_1,t)$ in (3).
In order to do that, we introduce the following new presentation of the operator $\hat{a}$,

$$b(k,1,t) = e^{i\omega^\lambda(k)t}\hat{a}(k,t), \qquad b(k,-1,t) = e^{-i\omega^\lambda(k)t}\hat{a}(-k,t)^*.$$

The following general equation for $b(k_1, \sigma, t)$ can then be derived,

$$
\begin{aligned}
\frac{\partial}{\partial t} b(k_1, \sigma, t) = & -i\sigma\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 - k_2 - k_3 - k_4) \\
& \times \frac{1}{2}\Big[(1+\sigma)\hat{V}(k_2 + k_3) + (1-\sigma)\hat{V}(k_3 + k_4)\Big] \\
& \times \exp\big[-it(-\sigma\omega^\lambda(k_1) - \omega^\lambda(k_2) + \sigma\omega^\lambda(k_3) + \omega^\lambda(k_4))\big] \\
& \times [\Phi_1(k_2, k_3, k_4) b(k_2, -1, t) b(k_3, \sigma, t) b(k_4, 1, t) \\
& \quad + \Phi_0(k_2, k_3, k_4)[b(k_2, -1, t) b(k_3, \sigma, t) b(k_4, 1, t)]^T]
\end{aligned}
$$

The following general equation for $b(k_1, \sigma, t)$ can then be derived,

$$\frac{\partial}{\partial t} b(k_1, \sigma, t) = -i\sigma\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 - k_2 - k_3 - k_4) \exp\left[-it\Theta(k_2, k_3, k_4, \sigma)\right]$$
$$\times \Big[\Psi_1(k_2, k_3, k_4, \sigma) b(k_2, -1, t) b(k_3, \sigma, t) b(k_4, 1, t)$$
$$+ \Psi_0(k_2, k_3, k_4, \sigma)[b(k_2, -1, t) b(k_3, \sigma, t) b(k_4, 1, t)]^T\Big].$$

The following general equation for $b(k_1, \sigma, t)$ can then be derived,

$$\frac{\partial}{\partial t} b(k_1, \sigma, t) = -i\sigma\lambda \iiint_{(\Omega^*)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 - k_2 - k_3 - k_4) \exp\left[-it\Theta(k_2, k_3, k_4, \sigma)\right]$$
$$\times \Big[\Psi_1(k_2, k_3, k_4, \sigma) b(k_2, -1, t) b(k_3, \sigma, t) b(k_4, 1, t)$$
$$+ \Psi_0(k_2, k_3, k_4, \sigma)[b(k_2, -1, t) b(k_3, \sigma, t) b(k_4, 1, t)]^T\Big].$$

Recall the chain rule

$$\frac{\partial}{\partial s} \prod_{j=1}^{n} b(k_j, \sigma_j, s) = \sum_{m=1}^{n} \left(\prod_{j=1}^{m-1} b(k_j, \sigma_j, s) \right) \frac{\partial}{\partial s} b(k_m, \sigma_m, s) \left(\prod_{j=m+1}^{n} b(k_j, \sigma_j, s) \right),$$

Introduce a $\gamma > 0$, differentiate the product $e^{\gamma s} \prod_{j=1}^{n} b(k_j, \sigma_j, s)$, and insert the expression for $\frac{\partial}{\partial s} b(k_m, \sigma_m, s)$. Integrating again over s from 0 to t, we get

$$
\begin{aligned}
\prod_{j=1}^{n} b(k_j,\sigma_j,t) &= e^{-\gamma t}\prod_{j=1}^{n} b(k_j,\sigma_j,0) + \gamma\int_0^t \mathrm{d}s e^{-(t-s)\gamma}\prod_{j=1}^{n} b(k_j,\sigma_j,s) \\
&- i\lambda\sum_{m=1}^{n}\sigma_m\int_0^t \mathrm{d}s\iiint_{(\Omega^*)^3}\mathrm{d}k_2'\mathrm{d}k_3'\mathrm{d}k_4'\delta(k_m-k_2'-k_3'-k_4') \\
&\times\exp\Big[-(t-s)\gamma - is\Theta(k_2',k_3',k_4',\sigma_m)\Big] \\
&\times\prod_{j=1}^{m-1} b(k_j,\sigma_j,s)\big[\Psi_1(k_2',k_3',k_4',\sigma_m)b(k_2',-1,s)b(k_3',\sigma_m,s)b(k_4',1,s) \\
&+ \Psi_0(k_2',k_3',k_4',\sigma_m)[b(k_2',-1,s)b(k_3',\sigma_m,s)b(k_4',1,s)]^T\big]\prod_{j=m+1}^{n} b(k_j,\sigma_j,s). \qquad (4)
\end{aligned}
$$

$$\prod_{j=1}^{n} b(k_j,\sigma_j,t) = e^{-\gamma t}\prod_{j=1}^{n} b(k_j,\sigma_j,0) + \gamma\int_0^t \mathrm{d}s e^{-(t-s)\gamma}\prod_{j=1}^{n} b(k_j,\sigma_j,s)$$
$$- i\lambda\sum_{m=1}^{n}\sigma_m\int_0^t \mathrm{d}s\iiint_{(\Omega^*)^3}\mathrm{d}k_2'\mathrm{d}k_3'\mathrm{d}k_4'\delta(k_m-k_2'-k_3'-k_4')$$
$$\times\exp\Big[-(t-s)\gamma - is\Theta(k_2',k_3',k_4',\sigma_m)\Big]$$
$$\times\prod_{j=1}^{m-1} b(k_j,\sigma_j,s)\big[\Psi_1(k_2',k_3',k_4',\sigma_m)b(k_2',-1,s)b(k_3',\sigma_m,s)b(k_4',1,s)$$
$$+\Psi_0(k_2',k_3',k_4',\sigma_m)[b(k_2',-1,s)b(k_3',\sigma_m,s)b(k_4',1,s)]^T\big]\prod_{j=m+1}^{n} b(k_j,\sigma_j,s). \tag{4}$$

Inductively plugging (4) back into itself at the terms containing Ψ_1 leads to an expansion

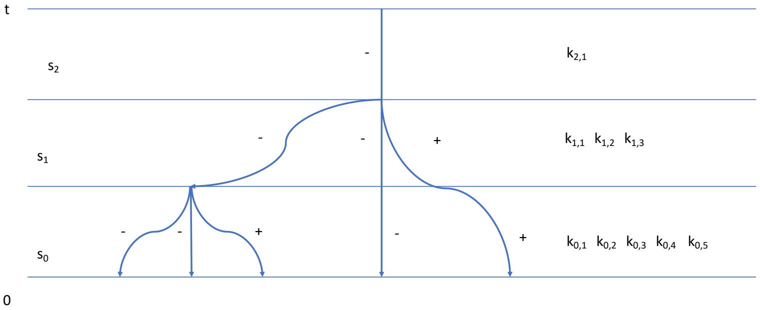
$$b(k,\sigma,t) = \sum_{n=0}^{N-1}(-i\lambda)^n\overbrace{\mathcal{E}_n^0(t,k,\sigma,\Gamma)[b(0)]}^{\text{Contains time-zero terms}} + \sum_{n=0}^{N-1}(-i\lambda)^n\gamma_n\int_0^t \mathrm{d}s_0\,\overbrace{\mathcal{E}_n^1(s_0,t,k,\sigma,\Gamma)[b(s_0)]}^{\text{Contains sub-leading }\Psi_1\text{-terms}}$$
$$+\sum_{n=1}^{N}(-i\lambda)^n\int_0^t \mathrm{d}s_0\,\underbrace{\mathcal{E}_n^2(s_0,t,k,\sigma,\Gamma)[b(s_0)]}_{\text{Contains truncated products}} + (-i\lambda)^N\int_0^t \mathrm{d}s_0\,\underbrace{\mathcal{E}_N^3(s_0,t,k,\sigma,\Gamma)[b(s_0)]}_{\text{Contains leading }\Psi_1\text{-terms}}.$$

Diagrammatic representation

$$
\begin{aligned}
b(k,\sigma,t) &= e^{-\gamma t} b(k,\sigma,0) + \gamma \int_0^t \mathrm{d}s e^{-(t-s)\gamma} b(k,\sigma,s) \\
&- i\lambda\sigma \int_0^t \mathrm{d}s \iiint_{(\Omega^*)^3} \mathrm{d}k_2' \mathrm{d}k_3' \mathrm{d}k_4' \delta(k - k_2' - k_3' - k_4') \exp\left[-(t-s)\gamma - is\Theta(\bar{k}',\sigma) \right] \\
&\times \left[\Psi_1(\bar{k}',\sigma) b(k_2',-1,s) b(k_3',\sigma,s) b(k_4',1,s) + \Psi_0(\bar{k}',\sigma)[b(k_2',-1,s) b(k_3',\sigma,s) b(k_4',1,s)]^T \right].
\end{aligned}
$$

Diagrammatic representation

$$b(k,\sigma,t) = e^{-\gamma t} b(k,\sigma,0) + \gamma \int_0^t \mathrm{d}s e^{-(t-s)\gamma} b(k,\sigma,s)$$
$$- i\lambda\sigma \int_0^t \mathrm{d}s \iiint_{(\Omega^*)^3} \mathrm{d}k_2'\mathrm{d}k_3'\mathrm{d}k_4' \delta(k - k_2' - k_3' - k_4') \exp\Big[-(t-s)\gamma - is\Theta(\bar{k}',\sigma) \Big]$$
$$\times \Big[\Psi_1(\bar{k}',\sigma) b(k_2',-1,s) b(k_3',\sigma,s) b(k_4',1,s) + \Psi_0(\bar{k}',\sigma) [b(k_2',-1,s) b(k_3',\sigma,s) b(k_4',1,s)]^T \Big].$$



Thank you for your attention!