The Free Energy Of Dilute Bose Gases At Low Temperatures

I Bose Gases

Fermions "Spin" $\in \frac{1}{2} + \mathbb{Z}$ • Proton, Neutron, Electron

Bosons "Spin" $\in \mathbb{Z}$ • Higgsboson, Photon

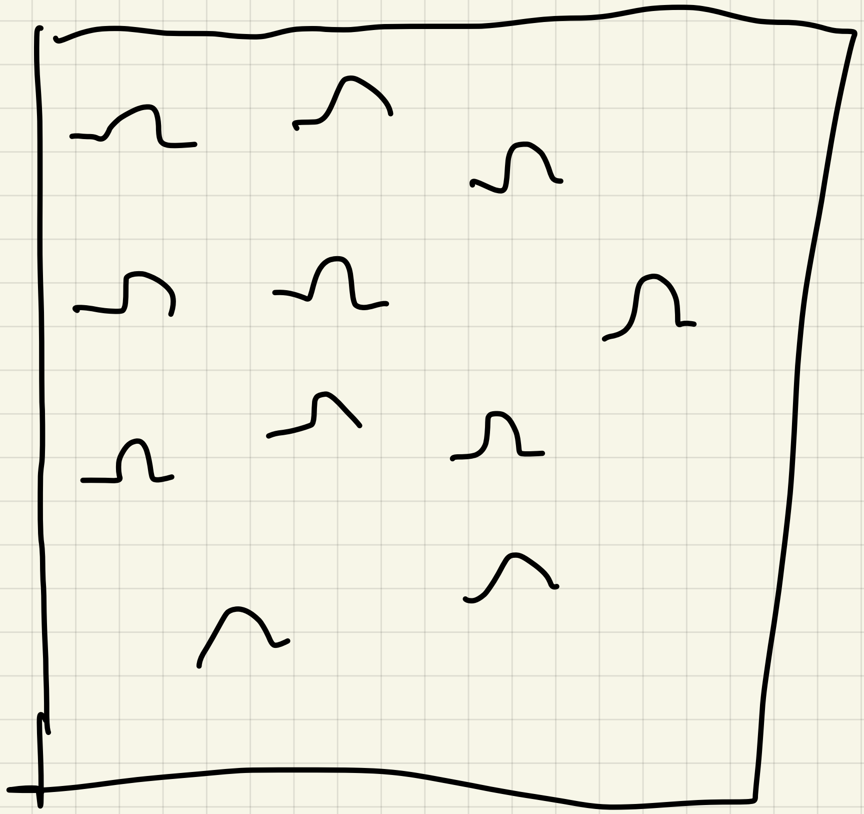
Atom/Molecules • Helium

e^-

(Superfluidity 1938 → Nobel prize 1978)

e^-

• Rubidium (BEC 1995 → Nobel prize 2001)



$\Lambda = [-\frac{L}{2}, \frac{L}{2}]^3 \subseteq \mathbb{R}^3$, a single boson is given by $\psi \in L^2(\Lambda)$

A system of N-bosons is given by

$\psi_N \in L^2_s(\Lambda^N)$

$$\psi(\ldots, x_i, \ldots, x_j, \ldots) = \psi(\ldots, x_j, \ldots, x_i, \ldots) \quad \forall i, j$$

The bosons interact

let $V \geq 0$

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} V(x_i - x_j)$$

II Bogoliubov Theory (1946)

We are interested in $\sigma(H_N)$. For this we use 2^{nd} quantization

$$H_N = \sum_{p \in \frac{2\pi}{L}\mathbb{Z}^3} p^2 a_p^\dagger a_p + \frac{1}{2L^3} \sum_{p,q,r \in \frac{2\pi}{L}\mathbb{Z}^3} \hat{V}(r) a_{p+r}^\dagger a_q^\dagger a_{q+r} a_p$$

$$\hat{V}(r) = \int_{\mathbb{R}^3} V(x) e^{ir \cdot x} dx$$

$a_p : L_s^2(\Lambda^N) \longrightarrow L_s^2(\Lambda^{N-1})$, $\quad (a_p \psi)(x_1, \dots, x_{N-1}) = \sqrt{N} \int_\Lambda \overline{\varphi_p(x_N)}\, \psi(x_1, \dots, x_N)\, dx_N$

$a_p^\dagger = a_p^*$

In particular $\mathcal{N} := \sum_p a_p^\dagger a_p$ is $\mathcal{N}\big|_{L_s^2(\Lambda^N)} = N$

Bogoliubov argues that since $\exists$ BEC

$$\underbrace{\#\text{ excited particles}}_{N_+ = \sum_{p\neq 0} a_p^\dagger a_p} \ll \underbrace{\#\text{ particles in ground state}}_{a_0^\dagger a_0 \sim N_0}$$

$\Rightarrow$ omit all terms with ≥ 3 a_p, $p \neq 0$

$$H_N = \sum_p p^2 a_p^\dagger a_p + \frac{1}{2L^3} \sum_{p,q,r} \hat{V}(r)\, a_{p+r}^\dagger a_q^\dagger a_{q+r} a_p$$

$$\overset{\text{Bog}}{\approx} \sum_{p\neq 0} p^2 a_p^\dagger a_p + \frac{1}{2L^3} \hat{V}(0) N_0^2 + 0 + \frac{N_0}{L^3} \hat{V}(0) \sum_{p\neq 0} a_p^\dagger a_p$$

$$+ \frac{N_0}{L^3} \sum_{p\neq 0} \hat{V}(p) a_p^\dagger a_p + \frac{N_0}{2L^3} \sum_{p\neq 0} \hat{V}(p) \left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p} \right)$$

$$N_0 = N - N_+ \quad \approx \frac{\hat{v}(0) N^2}{2L^3} + \sum_{p\neq 0}\left(p^2 + \frac{N}{L^3}\hat{v}(p)\right) a_p^\dagger a_p + \frac{1}{2}\sum_{p\neq 0}\frac{N}{L^3}\hat{v}(p)\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right)$$

$$\rho = \frac{N}{L^3} \quad = \rho N \frac{\hat{v}(0)}{2} + \frac{1}{2}\sum_{p\neq 0}\sqrt{p^4 + 2\rho\hat{v}(p)p^2} - p^2 - \rho\hat{v}(p)$$

$$+ \sum_{p\neq 0}\sqrt{p^4 + 2\rho\hat{v}(p)p^2}\; b_p^\dagger b_p$$

$$\boxed{b_p = \mathcal{V}^* a_p \mathcal{V}}$$

$$=: E_{Bog} + \underline{\sum_{p\neq 0} e_p b_p^\dagger b_p} =: H_{Bog}$$

$$\Rightarrow \sigma(H_{Bog}) = E_{Bog} + \left\{ \sum_{p\neq 0} e_p n_p \;\middle|\; \sum_{p\neq 0} n_p \leq N \right\}$$

$$\Rightarrow F_L(N,T) \geq E_{Bog} + T\sum_{p\neq 0} \log(1-e^{-e_p/T})$$

Thermodynamic Limit

$$f(\rho,T) = \lim_{\substack{L\to\infty \\ \rho=\frac{N}{L^3}}} \frac{F_L(N,T)}{L^3} \geq \lim_{L\to\infty} \frac{\rho^2}{2}\hat{V}(0) + \frac{1}{2L^3}\sum_{p\neq 0}\left[\sqrt{p^4+2\rho\hat{V}(p)p^2} - p^2 - \rho\hat{V}(p)\right]$$

$$+ T L^{-3}\sum_{p\neq 0}\log(1-e^{-e_p/T})$$

$$= \frac{\rho^2}{2}\hat{V}(0) + \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} \sqrt{p^4+2\rho\hat{V}(p)p^2} - p^2 - \rho\hat{V}(p)\, dp$$

$$+ \frac{T}{(2\pi)^3}\int_{\mathbb{R}^3} \log(1-e^{-e_p/T})\, dp$$

$$\approx \rho^2\left(\frac{\hat{V}(0)}{2} - \frac{1}{(2\pi)^3}\int \frac{\hat{V}(p)^2}{4p^2}\, dp\right) + \frac{T^{5/2}}{(2\pi)^3}\int_{\mathbb{R}^3} \log(1-e^{-\sqrt{p^4+2\rho\hat{V}(p)p^2}/T})$$

Thm (H. – Haintzl – Nam – Schlein – Seiringer – Triay 2023+)

Let $0 \le V(|x|) \in L^1$ cp supported, decreasing. Then for any $0 \le T \le \varrho a$ we have

$$f(\varrho, T) = \varrho^2 4\pi a + C_{LHY} (\varrho a)^{5/2} + \frac{T^{5/2}}{(2\pi)^3} \int \log\left(1 - e^{-\sqrt{p^4 + 16\pi a \varrho p^2}/T}\right) dp$$
$$+ o\left((\varrho a)^{5/2}\right)$$

in the "dilute limit" $\varrho a^3 \to 0$

Remarks: Scattering length $4\pi a = \frac{\hat{V}(0)}{2} - \langle \frac{V}{2}, \frac{1}{-\Delta + V/2} \frac{V}{2} \rangle$

$$= \frac{\hat{V}(0)}{2} - \langle \frac{1}{2} V, \frac{1}{-\Delta} \frac{V}{2} \rangle + \langle \frac{V}{2}, \frac{1}{-\Delta} \frac{V}{2} \frac{1}{-\Delta} \frac{V}{2} \rangle \pm \dots$$

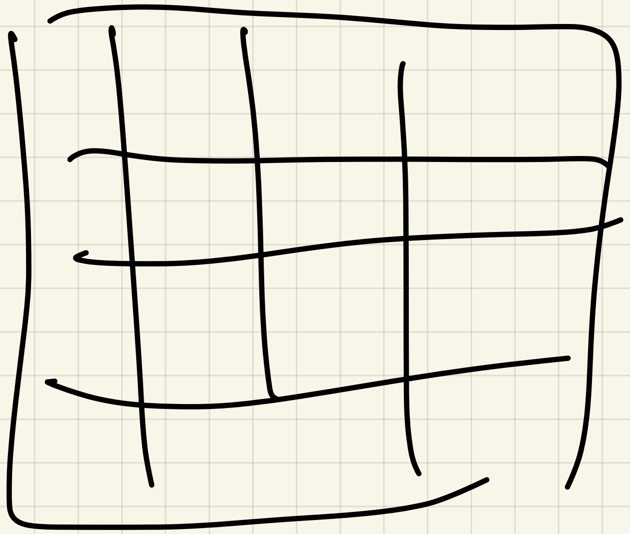
T=0 :

Upper bound: Dyson ('57), Yau–Yin ('09)

Lower bound: Lieb–Yngvason ('98), Fournais–Solovej (2020, 2022)

IV Proof

ℓ



L

$$\frac{F_L}{L^3} \geq \frac{F_\ell}{\ell^3}$$

Neumann ✓

$$\frac{F_L}{L^3} \leq \frac{F_\ell}{\ell^3}$$

Dirichlet (work in progress)

$$\frac{n^2}{\ell^7} \log \ell \ll \rho^{5/2}$$

$n = \rho \ell^3$

$$\Rightarrow \quad \ell = \rho^{-\frac{1}{2}-\kappa}, \quad \kappa > 0$$

Dirichlet: $\kappa > 1/2$