Hydrodynamic equations for integrable many-body systems

Project B3 inside the CRC TRR 352 «Mathematics of many-body quantum systems and their collective phenomena»

PHD candidate: A. Brollo

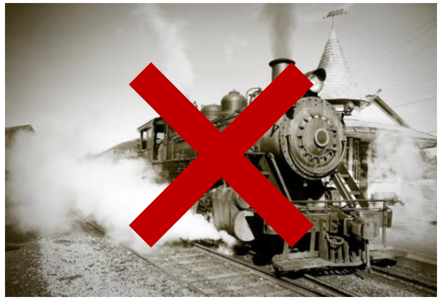
Supervisor: H. Spohn
Co-supervisor: C. Mendl

Herrsching Junior meeting, 13 December 2023

Outline

Hydrodynamics:

Water? 



- *Thermodynamics = set of laws that rule the equilibrium behaviour of macroscopic systems*

Hydrodynamics: large scale (macroscopic) behaviour of many-body systems out-of-equilibrium

- *Why?* *Thermodynamics is the most successfull theory in physics*

  ---> extremely wide range of applications

- Non equlibrium physics is way more complex, different approaches

  Hydrodynamics follows the path of thermodynamics and it's one of them

Outline

- Why hydrodynamics of integrable systems?

  …and what is an integrable system?

(later!)

- *∞-set of conserved quantities* ⇒
  - *extremely constrained dynamics*
  - *exactly solvable (non-perturbative)*

⇙

New paradigms and insight on hydrodynamics!

Outline

- Statistical Mechanics
  - Brief review of equlibrium statistical mechanics
  - Hydrodynamics in a nutshell

- Integrability
  - What is an integrable system?
  - A quantum Newton cradle: non-ergodicity

- Hydrodynamics of integrable systems:
  Generalized Hydrodynamics (GHD)

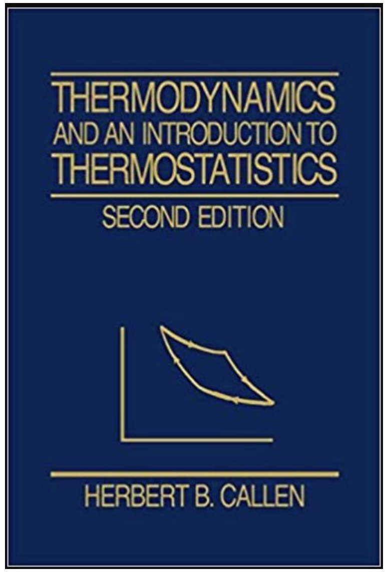
- Why does SFB pay me?

Brief review of statistical mechanics

*«Thermodynamics is not based on a new and particular law of nature.
It instead reflects a commonality or universality of all laws.*

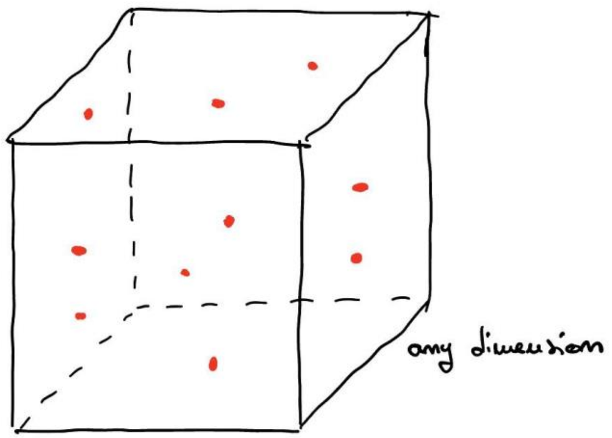
In brief, thermodynamics is the study of the restriction on the possible properties of matter that follow from the symmetry property of the fundamental law of physics»

H.B. Callen



At the core of the behaviour of matter there are SYMMETRIES and not INTERACTIONS

Brief review of statistical mechanics



N-body system,   N large

$H(\vec{q},\vec{p})$ Hamiltonian

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

Equations of motion to solve → *«exponential wall» of many-body physics*

We need a reduction of the degree of freedom!

Statistical mechanics	Let's put a measure on the phase space (or Hilbert space)

Brief review of statistical mechanics

probability measure ρ: *could be whaterver and evolve according to some equation* $\dot{\rho} = f(\rho, \partial_x \rho, \dots)$

Empirical observation: *macroscopic systems tend to reach an equilibrium configuration*

WHICH IS THE EQUILIBRIUM CONFIGURATION?

Let's put some formulas… (QUANTUM NOTATION, mutatis mutandis for classical)

$\mathcal{O}$ *local operator*

$\hat{\rho}$ *density matrix*

$$\langle \mathcal{O} \rangle_{\hat{\rho}} = \mathrm{Tr}[\hat{\rho}\mathcal{O}] \xrightarrow{t\to\infty} \mathrm{Tr}\left[e^{-\beta H}\mathcal{O}\right] = \langle \mathcal{O} \rangle_{\beta}$$

For some β=1/T

THERMALIZATION

Brief review of statistical mechanics

- Thermalization: *I know the density matrix at equlibrium of a many-body system*

$$\hat{\rho}_\beta = \frac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}$$

GIBBS STATE

Boltzmann 1868

$$\langle \mathcal{O} \rangle_\beta = \mathrm{Tr}[\hat{\rho}_\beta \mathcal{O}]$$

- *Can I only wait for the system to equilibrate?* SEPARATION OF SCALE

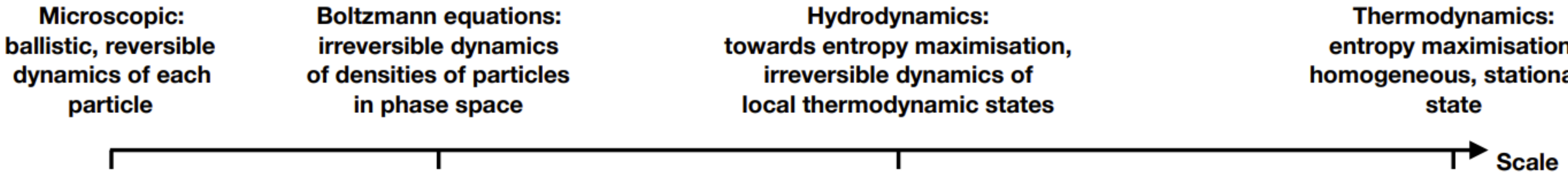
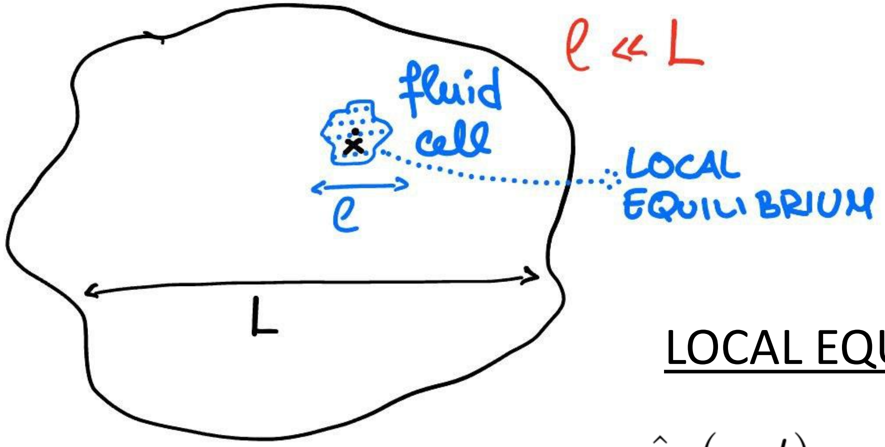


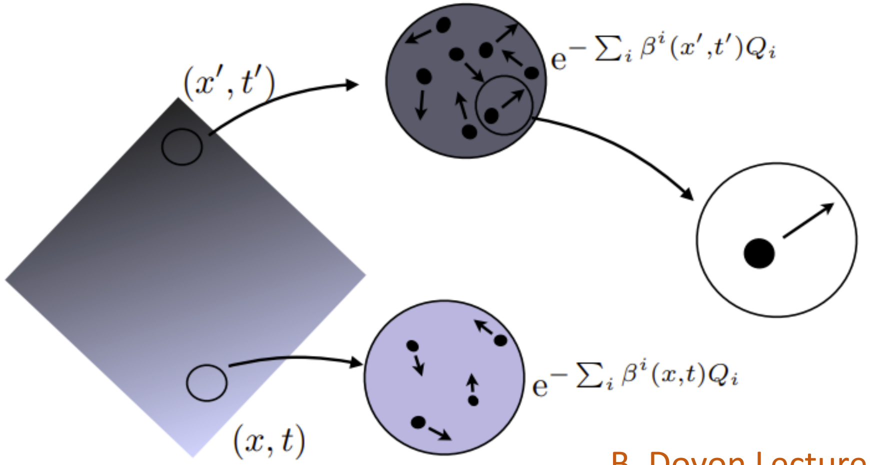
Figure 2.1: Time scales in gases with their various theoretical descriptions.

B. Doyon Lecture notes on GHD

Hydrodynamics in a nutshell

- Fluid cell: *introduce a mesoscopic scale*





B. Doyon Lecture notes on GHD

LOCAL EQUILIBRIUM

$$\hat{\rho}_\beta(x,t) \propto e^{-\beta(x,t)h(x)}$$

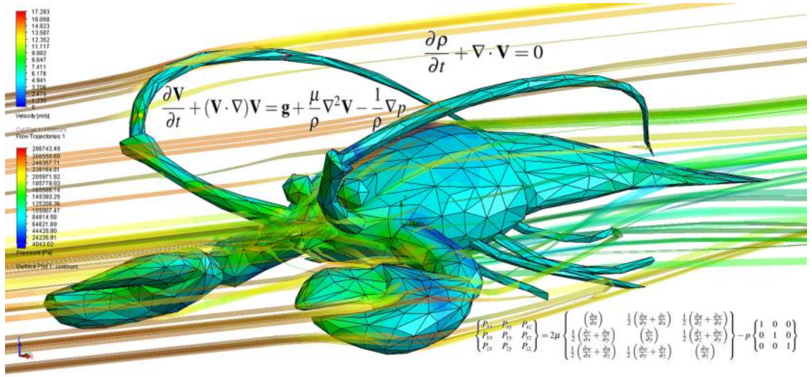
$$H = \int dx h(x)$$ $h(x)$ Hamiltonian density

- *Every fluid cell (labelled by x) has its own Gibbs state*

HYDRODYNAMIC APPROXIMATION

- *I need an equation for the evolution of* $\beta(x,t)$ *(PDE)*

Hydrodynamics in a nutshell



- Example: NAVIER-STOKES EQUATIONS

$$\partial_t \rho(x,t) + \partial_x [v(x,t)\rho(x,t)] = 0,$$

$$\partial_t v(x,t) + v(x,t)\partial_x v(x,t) = \frac{1}{\rho(x,t)}[-\partial_x P[\rho(x,t)] + \xi \partial_x^2 v(x,t)]$$

If I recollect some terms…

$$p = v\rho \qquad j_p = P + v^2\rho - \xi \partial_x v$$

…NS equations rewrite as

$$\partial_t \rho(x) + \partial_x j_\rho(x) = 0$$
$$\partial_t p(x) + \partial_x j_p(x) = 0$$

Conservation laws (continuity equation)

Conservation laws arise from the microscopic but hold at any scale!

Hydrodynamics in a nutshell

NOETHER

Spatial transl.	*Momentum P conserved*
Time transl.	*Energy H conserved*
Isolated syst.	*Mass M conserved*

Most common systems in nature: GALILEIAN SYSTEMS

These are all local conserved quantities → **CONSERVATION LAWS**

$$M = \int dx \rho(x) \quad P = \int dx p(x) \quad H = \int dx h(x)$$

$$\begin{cases} \partial_t \rho(x) + \partial_x j_\rho(x) = 0 \\ \partial_t p(x) + \partial_x j_p(x) = 0 \\ \partial_t h(x) + \partial_x j_h(x) = 0 \end{cases}$$

HYDRODYNAMICS: conservation laws of the fundamental microscopic model applied to quantities averaged on local equilibrium states

$$\mathrm{q}_i(x,t) = \langle q_i \rangle_{\beta(x,t)}$$

$$\partial_t \mathrm{q}_i(x,t) + \partial_x \mathrm{j}_i(x,t) = 0$$

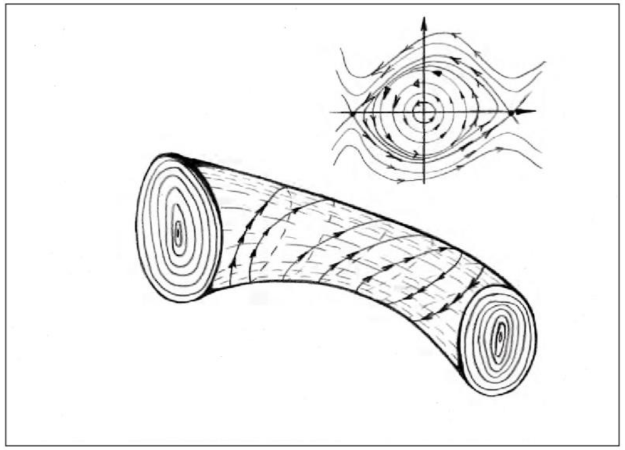
- Hydrodynamics

- Integrable systems

- Generalized Hydrodynamics

- My work at SFB

Integrability



- Classically: *Arnold-Liouville, existence of action-angle variables*
- Quantum: *not clear!*

Remarks on the notion of quantum integrability

Jean-Sébastien Caux and Jorn Mossel

Anyway, for our purposes it is sufficient to consider an integrable system as:

a system with an extensive (linear in the d.o.f) number of local conserved charges in involution

$$Q_i = \int dx q_i(x) \qquad i = 1, \ldots, N \qquad [Q_i, Q_j] = 0$$

Exactly solvable:

-quantum: Bethe ansatz

-classical: IST, scattering coordinates, …

Exactly solvable doesn't mean analytically, means with no approximation (pert. theory, mean field, ecc…)

Integrability

- Two scenarios in the landscape of models:

GALILEIAN SYSTEMS	3 cons charges:	Mass, Momentum, Energy
INTEGRABLE SYSTEMS	N cons. charges:	Mass, Momentum, Energy, more and more…

- Example? *-Free systems! N free particles of momentum* $p_i \; i = 1, \dots, N$

$$Q_n = \sum_i (p_i)^n$$ *are conserved*

Interacting? *QUANTUM: δ-Bose gas or Lieb-Liniger gas*

$$\mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<l} \delta(x_j - x_l),$$

Euler-Lagrange equation of the Lieb-Liniger model is the NLS

$$i\partial_t \psi = -\partial_x^2 \psi + \kappa |\psi|^2 \psi$$

Does integrability exist?

- Coleman-Mandula theorem: *for a relativistic (non-free) theory in d>1 there are no spacetime symmetries other then the Poincarè group => Impossible integrability* Except for conformal field theories

All Possible Symmetries of the S Matrix*

Sidney Coleman† and Jeffrey Mandula‡
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
(Received 16 March 1967)

- If we put ourself in (1+1) dimension, we break one of the hypothesis!

INTEGRABILITY IS A ONE-DIMENSIONAL STUFF!

Does integrability exist?

- Coleman-Mandula theorem: *for a relativistic (non-free) theory there are no spacetime symmetries other then the Poincarè group* => *Impossible integrability* Except for conformal field theories

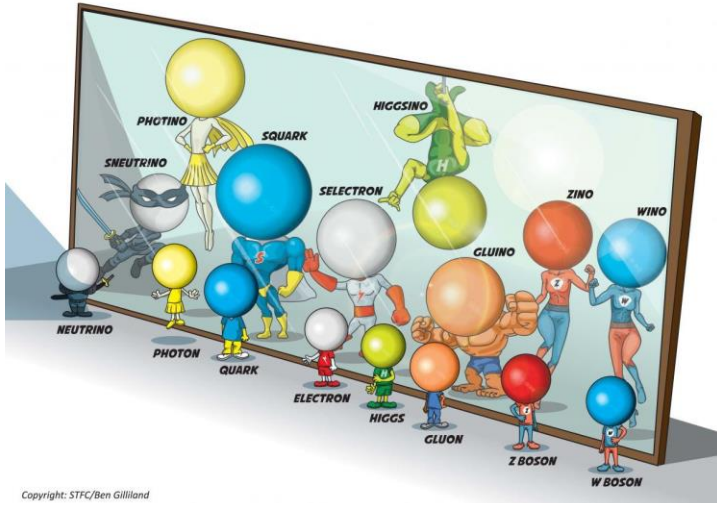
All Possible Symmetries of the *S* Matrix*

Sidney Coleman† and Jeffrey Mandula‡
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
(Received 16 March 1967)

- If we put ourself in (1+1) dimension, we break one of the hypothesis!

INTEGRABILITY IS A ONE-DIMENSIONAL STUFF!

- Beyond Coleman-Mandula theorem? Supersymmetry!



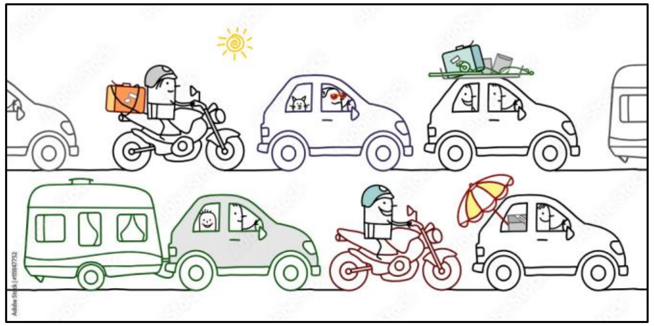
Why study the 1D world?

- What drives the research? <u>FUN!</u>

«However, my personal reason for working on one-dimensional problems is merely that they are fun. A man grows stale if he works all the time on the insoluble and a trip to the beautiful work of one dimension will refresh his imagination better than a dose of LSD»

Freeman Dyson, 1967

INDIVIDUAL BEHAVIOUR



COLLECTIVE BEHAVIOUR

- <u>Future outlooks</u>: electronics, cond mat physics, physics at the edge, …

- <u>Mathematical physics</u>: *one-dimensional physics is different!*

  - individual excitations -------> collective excitations

  - Coleman-Mermin-Wagner theorem: no long-range order, fluctuations have a preminant role

The issue of thermalization

A QUANTUM NEWTON CRADLE

A quantum Newton cradle

nature Vol 440|13 April 2006|doi:10.1038/nature04693

LETTERS

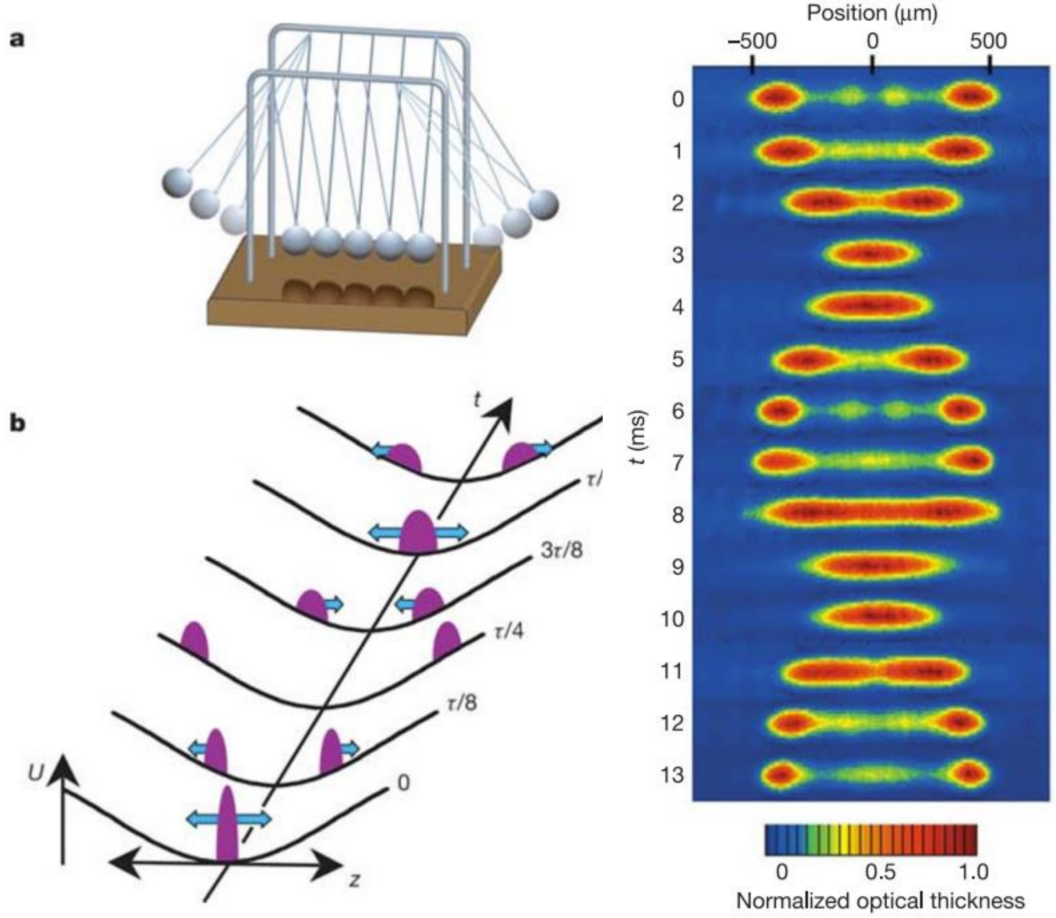
A quantum Newton's cradle

Toshiya Kinoshita[1], Trevor Wenger[1] & David S. Weiss[1]

- *Two blobs of low density Rb87 atoms*
  -described by Lieb-Liniger model

Integrable systems do not thermalize

Morally it's the «modern version of the Fermi-Pasta-Ulam problem»



A quantum Newton cradle

- *Toward what relaxes an integrable system?*

GALILEIAN SYSTEMS --------> GIBBS STATE $e^{-\beta(H-\mu N)}$

INTEGRABLE SYSTEMS --------> GENERALIZED GIBBS STATE

$$e^{-\sum_j^\infty \beta_j Q_j}$$

Nonequilibrium dynamics of closed interacting quantum systems.

Anatoli Polkovnikov[1], Krishnendu Sengupta[2], Alessandro Silva[3], Mukund Vengalattore[4]

Q_j *conserved quantities*

β_j *generalized inverse temperatures (or generalized chemical potentials)*

- Hydrodynamics
- Integrable systems
- Generalized Hydrodynamics
  (even though it is very specific)
- My work at SFB

Generalized Hydrodynamics (GHD)

CASTRO-ALVAREDO, DOYON, YOSHIMURA, PRX 6, 041065 (2016)

BERTINI, COLLURA, DE NARDIS, FAGOTTI, PRL 117, 207201 (2016)

…or better the hydrodynamics of integrable systems (iFluid?)

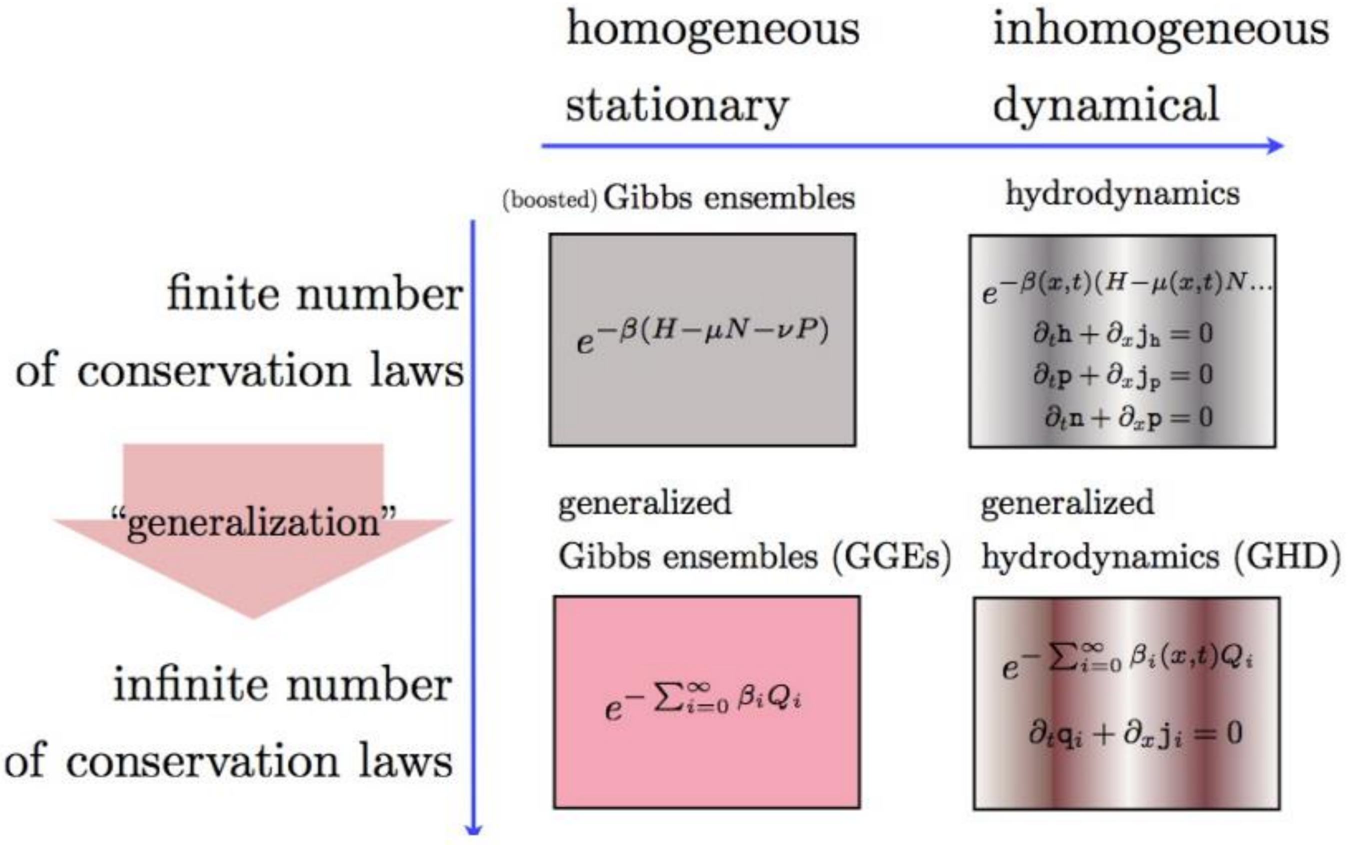


Figure 4.1: Where GHD fits.

B. Doyon Lecture notes on GHD

GHD EQUATIONS

$$\partial_t q_i(x,t) + \partial_x j_i(x,t) = 0$$

where

$$q_i(x,t) = \langle q_i \rangle_{\vec{\beta}(x,t)}$$

$$j_i(x,t) = \langle j_i \rangle_{\vec{\beta}(x,t)}$$

? What are the currents?

Generalized Hydrodynamics (GHD)

- *I need to close the equations…*

$$\vec{\beta}(x,t) \xrightarrow{\text{change of coords}} \langle q_i \rangle_{\vec{\beta}(x,t)} \quad \Rightarrow \quad \langle j_i \rangle_{\vec{\beta}(x,t)} = j_i(x,t)[q(x,t)]$$

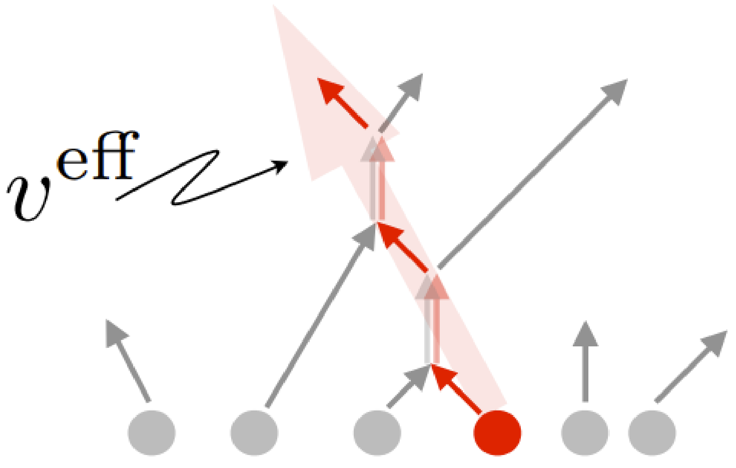
- *Now GHD equations are closed* $$\partial_t q_i(x,t) + \partial_x j_i(x,t)[q(x,t)] = 0$$

- *For the moment everything formal, but in practise what are the currents?*

Generalized Hydrodynamics (GHD)

- *GHD equations can be rewritten as*

$$\partial_t n(k,x,t) + v^{\text{eff}}(k,x,t)\partial_x n(k,x,t) = 0$$



- Collision rate ansatz

$$v^{\text{eff}}(k) = k + \int dk' \mathcal{K}(k-k')n(k')(v^{\text{eff}}(k) - v^{\text{eff}}(k'))$$

Now we figured out everything, we can do simulations, prediction, both analytical and numerical

prediction, both analytical and numerical

PHYSICAL REVIEW LETTERS **122,** 090601 (2019)

Editors' Suggestion | Featured in Physics

Generalized Hydrodynamics on an Atom Chip

M. Schemmer,[1] I. Bouchoule,[1] B. Doyon,[2] and J. Dubail[3]

[1]*Laboratoire Charles Fabry, Institut d'Optique, CNRS, Université Paris-Saclay, 91127 Palaiseau cedex, France*
[2]*Department of Mathematics, King's College London, Strand, London WC2R 2LS, United Kingdom*
[3]*Laboratoire de Physique et Chimie Théoriques, CNRS, Université de Lorraine, UMR 7019, F-54506 Vandoeuvre-les-Nancy, France*

(Received 24 October 2018; revised manuscript received 12 December 2018; published 5 March 2019)

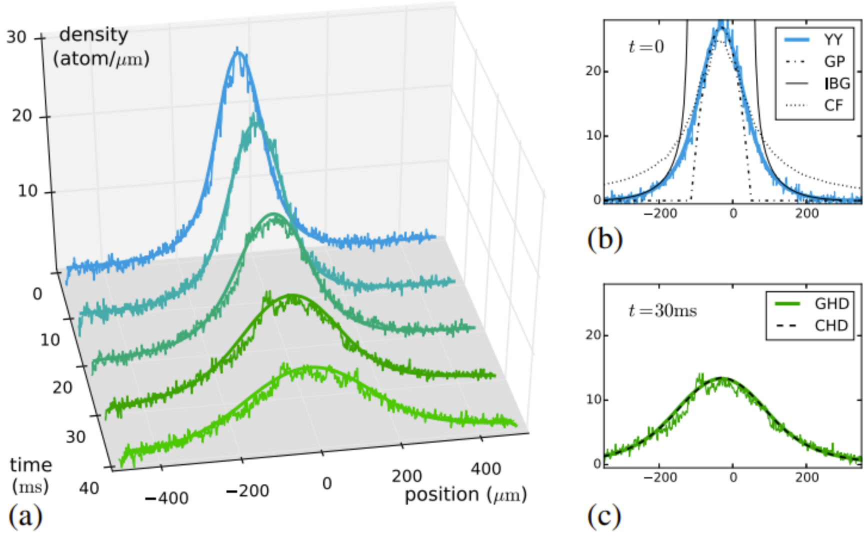


FIG. 1. (i) *In situ* density profile after longitudinal expansion from a harmonic trap of a 1D cloud of $N = 4600 \pm 100$ ^{87}Rb atoms; the smooth curve is the theoretical prediction of GHD and the noisy one is the experimental data. (ii) Initial profile obtained from the Yang-Yang equation of state (YY), Gross-Pitaevskii, ideal Bose gas, and classical field [40], with the same temperature and chemical potential as for YY. (iii) Evolution from the YY initial profile with GHD and conventional hydrodynamics.

QUANTUM GASES

Generalized hydrodynamics in strongly interacting 1D Bose gases

Neel Malvania[1]†, Yicheng Zhang[1]†, Yuan Le[1], Jerome Dubail[2], Marcos Rigol[1], David S. Weiss[1]*

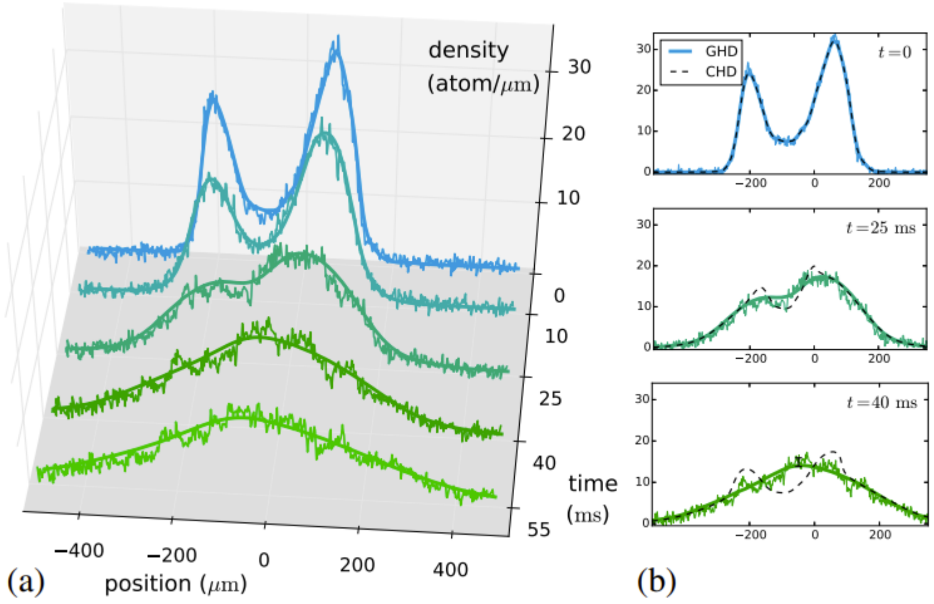


FIG. 3. (i) Longitudinal expansion of a cloud of $N = 6300 \pm 200$ atoms initially trapped in a double-well potential, compared with GHD. (ii) Even though the initial state is the same for GHD and CHD, both theories clearly differ at later times. CHD wrongly predicts the formation of two large density waves. The error bar shown at the center at $t = 40$ ms corresponds to a 68% confidence interval, and is representative for all data sets.

- Hydrodynamics ✓
- Integrable systems ✓
- Generalized Hydrodynamics ✓
  (even though it is very specific)
- My work at SFB ←

My work at SFB

The field is very recent and research is moving fast and in several directions

Emergent Hydrodynamics in Integrable Many-Body Systems,
A. Bastianello (TUM physics), B. Doyon, B. Bertini, R. Vasseur

- Mathematical foundation:
  - *Derivation of the equations*
  - *Definition of the currents* Spohn 20, B. Poszgay 21
  - *Weak integrability breaking* A. Bastianello
  - *Connection with soliton gases* G. El
  - *ecc…*

- Technological applications:
  - *Heat engines* I am doing a joint work with A. Bastianello
  - *more more and more*

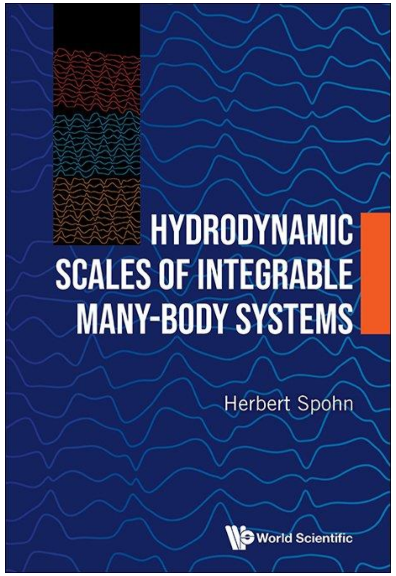
- Paradigms of non equilibrium physics:
  - *transport phenomena* Jacopo de Nardis
  - *ecc…*

My work at SFB

In my last (and first) work…

- «On a hydrodynamic scale all integrable many-particle systems are structurally alike »

  «Given the diversity of microscopic models such a claim is surprisingly bold»



- *All quantum models are «particle based» and you can solve them using the Bethe ansatz*

- *Classical models instead could be gases (particle), wave equations (solitons) or chains*

  ---> not trivial to establish properly why such an universal behaviour

My work at SFB

In my last (and first) work…

Ablowitz-Ladik chain: integrable discretization of NLS

- *We know how to derive its hydrodynamics in a statistical mechanical fashion*

Spohn 22

- *New: we derive a particle interpretation for the model reconstructing what are the quasi-particle* AB, Spohn to appear

- *Can we extend our results to the continuum limit (NLS)?*
- *Can we find a proper derivation for the currents?*
- *Can we extend our procedure to other models?*
- …

THANKS FOR THE ATTENTION!