Macroscopic Thermalization and the ETH for Highly Degenerate Hamiltonians

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Macroscopic thermal equilibrium

Let $\mathcal{H}$ be a finite dimensional Hilbert space, $\mathcal{H}_{\mathrm{eq}} \subset \mathcal{H}$ a subspace defining thermal equilibrium, and P_{eq} the orthogonal projection onto $\mathcal{H}_{\mathrm{eq}}$. It is said that a pure state $\psi \in \mathbb{S}(\mathcal{H})$ is in

macroscopic thermal equilibrium (MATE), if for some $\varepsilon \ll 1$

$$\psi \in \mathrm{MATE}_\varepsilon := \{\psi \in \mathbb{S}(\mathcal{H}) \,|\, \|P_{\mathrm{eq}}\psi\|^2 \geq 1-\varepsilon\}\,.$$

Example: For a system of particles in a box, one should think of $\mathcal{H}$ as a micro-canonical energy subspace $\mathcal{H} = \chi_{[E-\Delta,E]}(H)$, where H is the Hamiltonian of the system and $\Delta > 0$ is small but still large enough so that dim$\mathcal{H}$ is huge. A natural choice for $\mathcal{H}_{\mathrm{eq}}$ would be a subspace of states ψ for which the coarse-grained position distribution is close to uniform (see below) and the coarse-grained momentum distribution is close to Maxwellian with the appropriate temperature. See Figure 1 for a screenshot of a simulation that illustrates the corresponding classical situation.

ETH and approach to MATE

We say that a Hamiltonian H satisfies the

eigenstate thermalization hypothesis (ETH), if

$\forall$ normalized eigenvector ϕ of H : $\quad \phi \in \mathrm{MATE}_\varepsilon\,.$

Theorem 1 [1]: Let H satisfies the ETH. Then for every $\psi_0 \in \mathbb{S}(\mathcal{H})$ and $(1-\sqrt{\varepsilon})$-most $t \in [0,\infty)$

$$\psi_t \in \mathrm{MATE}_{\sqrt{\varepsilon}}\,.$$

I.e., all initial states ψ_0 eventually evolve into $\mathrm{MATE}_{\sqrt{\varepsilon}}$ and stay there for most of the time. In short, we say that all initial states thermalize.

Problem of high degeneracies

Let D_E be the maximal degeneracy of a Hamiltonian H_0. If one eigenbasis $(\phi_\alpha)_\alpha$ of H_0 satisfies the ETH, then any normalized eigenvector ϕ of H_0 satisfies

$$\phi \in \mathrm{MATE}_{\varepsilon D_E}$$

and this bound can not be improved in general. Hence, for highly degenerate Hamiltonians ETH for one eigenbasis does not imply ETH for all eigenvectors. This is, e.g., the case for the free Fermi gas, where $D_E \geq 2^{dN}$.

Small perturbations of degenerate Hamiltonians

Theorem 2 [1]: Assume that H_0 has an eigenbasis $(\phi_\alpha)_\alpha$ that satisfies the ETH for some $\varepsilon > 0$, i.e.

$$\forall\alpha \;:\; \phi_\alpha \in \mathrm{MATE}_\varepsilon.$$

Let V be a self-adjoint perturbation drawn randomly from a continuous distribution that is invariant under conjugation with all unitaries commuting with H_0 and let for $\lambda \in \mathbb{R}$

$$H := H_0 + \lambda V\,.$$

Finally, let $\mathcal{H}_\nu \subset \mathcal{H}$ be any subspace (e.g. a subspace associated with a macroscopic non-equilibrium state).

Then for all $\delta, \delta'\delta'' > 0$ there exists $\lambda_0 > 0$ such that for all $\lambda \in (0,\lambda_0)$ and for $(1-\delta)$-most V, $(1-\delta')$-most $\psi_0 \in \mathbb{S}(\mathcal{H}_\nu)$ are such that for $(1-\delta'')$-most $t \in [0,\infty)$

$$\psi_t := \mathrm{e}^{-\mathrm{i}Ht}\psi_0 \in \mathrm{MATE}_{\varepsilon'}$$

with

$$\varepsilon' = \frac{3\varepsilon}{\delta\delta'\delta''}\,.$$

Note that H has non-degenerate eigenvalues with probability one, but its eigenbasis need not satisfy the ETH.

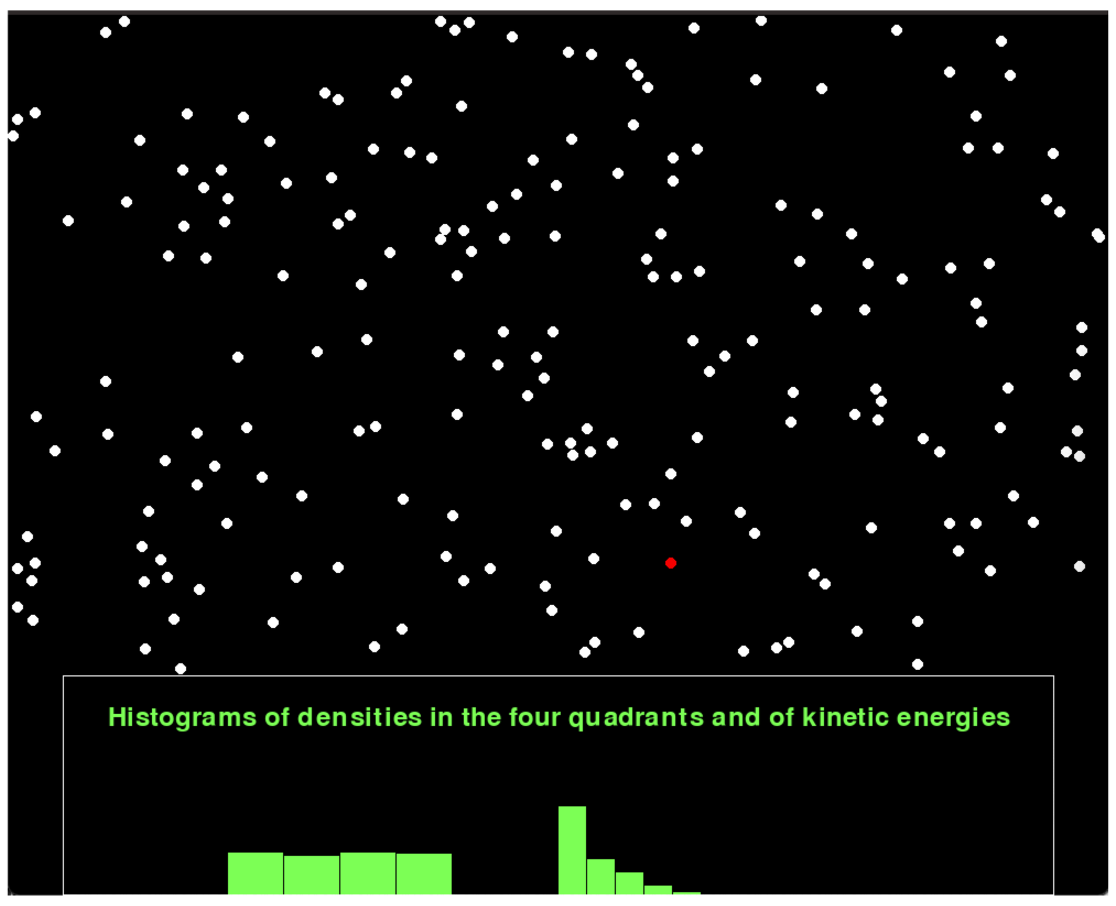

Figure 1. Classical gas in macroscopic thermal equilibrium

Applications

Free fermions in a box

Consider the free Fermi gas of N particles on a d-dimensional lattice $\Lambda = \{1,\ldots,L\}^d$ with periodic boundary conditions, i.e.

$$H_0 = -\sum_{\substack{x,y\in\Lambda\\ \mathrm{dist}(x,y)=1}} c_x^* c_y$$

restricted to the N-particle sector of Fock space.

For $\Gamma \subset \Lambda$ (e.g. one half of the box) let

$$N_\Gamma := \sum_{x\in\Gamma} c_x^* c_x\,,$$

$\mu := \frac{|\Gamma|}{|\Lambda|}$, and define for some $\alpha > 0$

$$P_{\mathrm{eq}} := \chi_{[\mu-\alpha,\mu+\alpha]}(N_\Gamma/N)\,.$$

For $d=1$ we show [1] that H_0 satisfies the ETH with

$$\varepsilon = \frac{32\ln N}{\alpha^2 N}.$$

Thus, by Theorem 1, for large N all initial states "thermalize" in the sense that for most times $\psi_t := \mathrm{e}^{-\mathrm{i}H_0 t}\psi_0$ displays a homogeneous coarse grained distribution in space.

For $d \geq 1$ we show [1] (based on a result by Tasaki for $d=1$ [2]) that there exists one eigenbasis (Slater-determinants of one-body eigenfunctions) for which the ETH is satisfied with

$$\varepsilon = 2\mathrm{e}^{-\frac{\alpha^2}{3\mu(1-\mu)}N}\,.$$

Thus, by Theorem 2, for most small perturbations V most initial states (even when conditioned on a small non-equilibrium subspace) "thermalize" in the sense that for most times $\psi_t := \mathrm{e}^{-\mathrm{i}(H_0+V)t}\psi_0$ displays a homogeneous coarse grained distribution in position space.

$2d$ Ising model

Tasaki [3] applied our Theorem 2 to the Ising model in $d=2$ below the critical temperature to prove that for all initial states in an appropriate energy shell most of the time (under perturbed evolution) the magnetization density is close to the equilibrium value, i.e. to the spontaneous magnetization given by the microcanonical expectation value. In this example it is explicit that some but not all eigenstates satisfy the ETH.

References

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[2] Hal Tasaki.
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