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The complexity of approximate Gottesman-Kitaev-Preskill states

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Setting

Goal: Prepare a bosonic state close in L^1-distance to a target state $|\Psi_{\text{target}}\rangle \in L^2(\mathbb{R})$.
Allowed operations:

(a) Preparation of the vacuum state $|\text{vac}\rangle$ and of the single qubit state $|0\rangle$.
(b) Single- and two-qubit unitaries
(c) Gaussian one-and two-mode unitaries of bounded strength
(d) (Qubit-controlled) single-mode phase space displacements of bounded strength
(e) Homodyne measurements and qubit measurements in the computational basis

Complexity of a bosonic state

We consider state preparation protocols with initial state $|\Psi\rangle = |\text{vac}\rangle^{\otimes(m+1)} \otimes |0\rangle^{\otimes m'}$.

2.1 Unitary state preparation: Given $\varepsilon > 0$ we consider a unitary U composed of gates from (b)-(d) acting on the staten$|\Psi\rangle$ such that

$$\left\|\text{tr}_{m,m'} U|\Psi\rangle\langle\Psi| U^\dagger - |\Psi_{\text{target}}\rangle\langle\Psi_{\text{target}}|\right\|_1 \leq \varepsilon\,.$$

2.2 Heralded state preparation: Given $\varepsilon > 0$ and $p \in (0,1]$, we consider protocols of the following form:

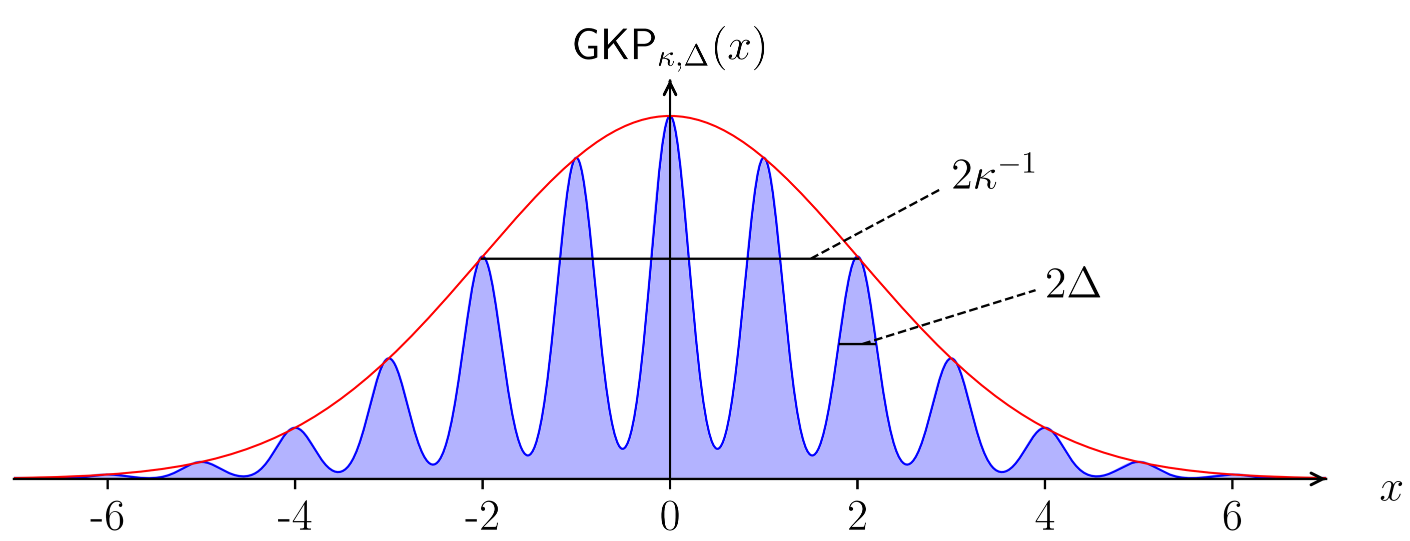
(i) Apply a unitary U composed of gates from (b)-(d) on $|\Psi\rangle$.
(ii) Measure the m auxiliary modes and m' qubits.
(iii) Depending on the measurement outcome, accept or reject.
(iv) Conditioned on acceptance apply a (mmt. dep.) displacement on the first mode.

Assume the protocol **accepts with probability at least** p and the **average state upon acceptance** ρ_{acc} satisfies

$$\|\rho_{\text{acc}} - |\Psi_{\text{target}}\rangle\langle\Psi_{\text{target}}|\|_1 \leq \varepsilon\,.$$

We define the **unitary state complexity** $\mathcal{C}^*_\varepsilon(|\Psi_{\text{target}}\rangle)$ and the **heralded state complexity** $\mathcal{C}^{*,\text{her}}_{p,\varepsilon}(|\Psi_{\text{target}}\rangle)$, respectively, as the **minimal number of operations** needed to prepare a state which is ε-close in L^1-distance to $|\Psi_{\text{target}}\rangle$.

Approximate GKP states



Gottesman-Kitaev-Preskill (GKP) states [2] are bosonic states first introduced in the context of **quantum fault-tolerance**. They act as a substrate to protect quantum information from phase space displacement noise. The ideal but unphysical GKP state is defined as the state stabilzed by the unitaries $S_P = e^{-iP}$ and $S_Q = e^{2\pi i Q}$. Formally, it is represented in position space as

$$|\text{GKP}\rangle \propto \sum_{z\in\mathbb{Z}} |z\rangle\,.$$

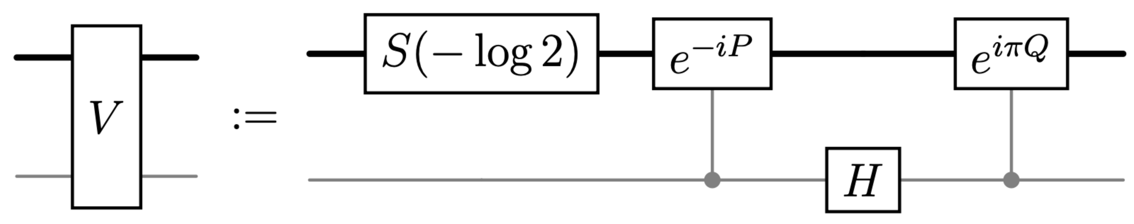
We are interested in approximate GKP states defined as

$$\text{GKP}_{\kappa,\Delta}(x) \propto \sum_{z\in\mathbb{Z}} e^{-\kappa^2 z^2/2} e^{-(x-z)^2/(2\Delta^2)}.$$

Efficient GKP state preparation

We present a heralded protocol that prepares an L^1-norm approximation of $|\text{GKP}_{\kappa,\Delta}\rangle$. It proceeds in two steps:

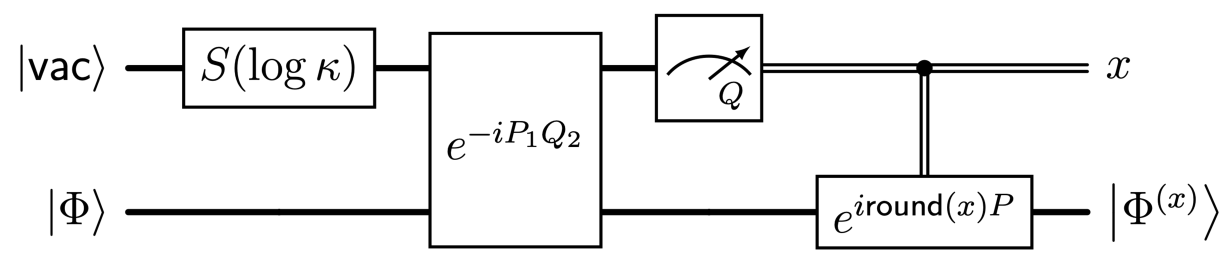
(I) **Comb state preparation** The comb state $\text{Ш}_{L,\Delta}(x) \propto \sum_{z=-L/2}^{L/2-1} e^{-(x-z)^2/(2\Delta^2)}$ is prepared by repeated application of the unitary V, in total $n = O(\log 1/\kappa)$-times,



which doubles the peaks with each application.

$$V\,(|\text{Ш}_{L,\Delta}\rangle \otimes |+\rangle) \approx |\text{Ш}_{2L,2\Delta}\rangle \otimes |+\rangle$$

(II) **Envelope Gaussification** A squeezed vacuum state $|\eta_\kappa\rangle = S(\log\kappa)\,|\text{vac}\rangle$ with variance κ^{-2} equips the comb state with a Gaussian envelope. Upon measuring x in an acceptance region, a classically controlled shift correction is applied.



Theorem 1 ([1])

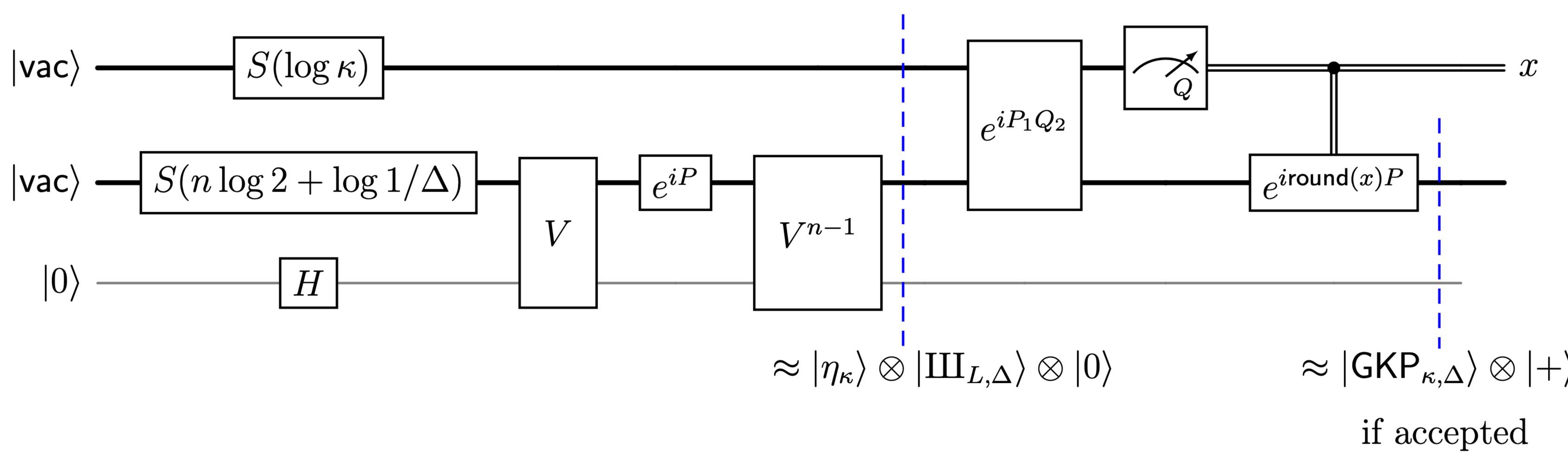
Let $\kappa, \Delta > 0$ sufficiently small. The heralded state preparation protocol described above prepares with probability at least $\Pr[\text{success}] > 1/10$ a quantum state ρ such that

$$\|\rho - |\text{GKP}_{\kappa,\Delta}\rangle\langle\text{GKP}_{\kappa,\Delta}|\|_1 \leq O(\sqrt{\Delta}) + O(\kappa^{1/3})$$

using $O(\log 1/\Delta + \log 1/\kappa)$ operations from (a)-(e).
In particular, there is a polynomial $\varepsilon(\kappa,\Delta)$ such that

$$\mathcal{C}^*_{1/10,\varepsilon(\kappa,\Delta)}(|\text{GKP}_{\kappa,\Delta}\rangle) \leq O(\log 1/\kappa + \log 1/\Delta) \quad \text{for} \quad (\kappa,\Delta) \to (0,0)\,.$$

The full circuit of the GKP state preparation protocol looks as follows



Lower bounds on the complexity

We also prove lower bounds on the unitary and heralded state complexity of $|\text{GKP}_{\kappa,\Delta}\rangle$.

Theorem 2 ([1])

Let $\kappa, \Delta > 0$. Then there is a polynomial $p(\kappa,\Delta)$ with $p(0,0) = 0$ such that

$$\mathcal{C}^*_1(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \Omega(\log 1/\kappa + \log 1/\Delta)$$
$$\mathcal{C}^{*,\text{her}}_{p,3p/2}(|\text{GKP}_{\kappa,\Delta}\rangle) \geq \Omega(\log 1/\kappa + \log 1/\Delta) \quad \text{for} \quad (\kappa,\Delta) \to (0,0)\,,$$

whenever $p \geq p(\kappa,\Delta)$.

Idea: The unitaries from (b)-(d) are **moment limited**. Thus, the energy after applying a circuit can grow at most exponentially in the circuit depth. This lower bounds the projection $\Pi_{[-R,R]}$ ($\widehat{\Pi}_{[-R,R]}$) of the output state in position (momentum) space. We infer that the distance to $|\text{GKP}_{\kappa,\Delta}\rangle$ is lower bounded depending on the circuit depth.

References

[1] L. Brenner, L. Caha, X. Coiteux-Roy, and R. Koenig.
The complexity of approximate Gottesman-Kitaev-Preskill states, 2024.
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[2] Daniel Gottesman, Alexei Kitaev, and John Preskill.
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