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B1: Norm Approximation for Interacting Many-Body Fermi Gases

Quantum Fluctuations of Many-Body Dynamics around the Gross-Pitaevskii Equation

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Abstract

We consider the evolution of a gas of N bosons in the three-dimensional Gross-Pitaevskii regime in which particles are initially trapped in a volume of order one and in which their dynamics is governed by the Gross-Pitaevskii Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{i<j}^{N} N^2 V(N(x_i - x_j)).$$

We construct a quasi-free approximation of the many-body dynamics, whose distance to the solution of the Schrödinger equation converges to zero, as $N \to \infty$, in the $L^2(\mathbb{R}^{3N})$-norm. To achieve this goal, we let the Bose-Einstein condensate evolve according to a time-dependent Gross-Pitaevskii equation. After factoring out the microscopic correlation structure, the evolution of the orthogonal excitations of the condensate is governed instead by a Bogoliubov dynamics, with a time-dependent generator quadratic in creation and annihilation operators. As an application, we show a central limit theorem for fluctuations of bounded observables around their expectation with respect to the Gross-Pitaevskii dynamics.

The Scattering Solution

- Two particle correlations can be described via the scattering solution: $\left[-\Delta + \frac{1}{2}V\right] f = 0$
- Outside of the support of V we have $f(x) = 1 - \frac{\mathfrak{a}}{|x|}$
- The scattering length $\mathfrak{a}$ is the effective range of the potential
- Truncation: Neumann problem on ball $|x| \leqslant \ell$:

$$\left[-\Delta + \frac{N^2}{2} V(Nx)\right] f_\ell(Nx) = N^2 \lambda_\ell f_\ell(Nx)$$

References

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Bose-Einstein Condensation

A sequence Ψ_N of N-particle wave functions exhibits complete Bose-Einstein condensation in $\varphi \in L^2(\mathbb{R}^3)$ if almost all particles are in this one-particle state. Mathematically, this means that

$$\lim_{N\to\infty} \frac{1}{N} \langle \psi_N, \sum_{i=1}^{N} (\mathbb{1} - |\varphi\rangle\langle\varphi|)_i \, \psi_N \rangle = 0 \quad (1)$$

where i specifies on which particle the operator acts. For minimizers of $H_N + \sum_{i=1}^{N} V_{ext}(x_i)$ this was shown in [6].

Condensate Evolution

After switching off the trap, the condensate wave function changes over time according to the time-dependent GP-equation (see [4]):

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi\mathfrak{a}|\varphi_t|^2\varphi_t, \quad \varphi_0 = \varphi. \quad (2)$$

The number of particles that is not in the condensate φ_t can also be bounded quantitatively. For initially trapped systems, it is of order 1 in N; in other words, the fraction appearing in (1) is $O(1/N)$ [3].

Beyond Condensate Evolution: Flucuation Dynamics

Most particles are in the condensate, and therefore, their evolution is described by (2). In order to describe the remaining fluctuations, it is useful to factor out the condensate.

- Decompose $\psi_{N,t} = \sum_{j=0}^{N} \psi_{N,t}^{(j)} \otimes_s \varphi_t^{\otimes(N-j)}$ where $\psi_{N,t}^{(j)} \in L^2_{\perp\varphi_t}(\mathbb{R}^3)^{\otimes_s j}$
- Define unitary $U_{N,t}\psi_{N,t} = \{\psi_{N,t}^{(0)}, \dots, \psi_{N,t}^{(N)}\} = \xi_{N,U_t} \in \mathcal{F}^{\leqslant N}_{\perp\varphi_t} = \bigoplus_{n=0}^{N} L^2_{\perp\varphi_t}(\mathbb{R}^3)^{\otimes_s n}$
- Fluctuation dynamics $i\partial_t \xi_{N,U_t} = (U_{N,t} H_N U_{N,t}^* + (i\partial_t U_{N,t}) U_{N,t}^*)\xi_{N,U_t}$
- This idea was introduced in [5] in order to obtain a norm approximation in the less singular mean field scaling where the interaction has the form $N^{-1}V$ and the condensate evolves as $i\partial_t\varphi_t = -\Delta\varphi_t + V * |\varphi_t|^2\varphi_t$. The norm approximation is obtained by comparing this evolution to one defined via a quadratic generator which gives rise to a time-dependent Bogoliubov transformation.

Bogoliubov Transformations

On the Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3)^{\otimes_s n}$ a unitary operator $\mathcal{U}\colon \mathcal{F} \to \mathcal{F}$ is a Bogoliubov transformation if its action on annihilation/creation operators is given as follows:

$$\mathcal{U}^* a(f) \mathcal{U} = a(Uf) + a^*(\overline{Vf})$$

for all $f \in L^2(\mathbb{R}^3)$ and bounded linear maps $U, V\colon L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ satisfying $U^*U - V^*V = \mathbb{1}$ and $U^*\bar{V} = V^*\bar{U}$.

Describing pair correlations

The most relevant correlations are pair correlations that can be be described via an (approximate) Bogoliubov transformation e^{B_t} for

$$B_t \approx -\frac{N}{2} \int [1 - f_\ell(N(x-y))] \varphi_t(x)\varphi_t(y) a_x^* a_y^* - \text{h.c.}$$

Factoring out these correlations allowed to prove norm approximations for potentials of the form $N^{3\beta-1}V(N^\beta \cdot)$ for $\beta < 1$ in [1, 2].

New Result: Quasi-Free Approximation in Gross-Pitaevskii Regime

Let $0 \leqslant V \in L^3(\mathbb{R}^3)$ be radial and compactly supported. Let $\varphi \in H^6(\mathbb{R}^3)$ and e^{B_t} as above. For $\psi_N \in L^2(\mathbb{R}^3)^{\otimes_s N}$ s.t.

$$\langle e^{-B_0} U_{N,0}\psi_N, (\mathcal{K}^2 + \mathcal{N}^6)\, e^{-B_0} U_{N,0}\psi_N \rangle \leqslant C,$$

where $\mathcal{N} = \int a_x^* a_x$ and $\mathcal{K} = \int \nabla_x a_x^* \nabla_x a_x$, holds uniformly in N there are $C, c > 0$ and $\omega_{N,t} \in \mathbb{R}$ s.t.

$$\| U_{N,t}\psi_{N,t} - e^{i\omega_{N,t}} e^{B_t} \mathcal{U}_2(t) e^{-B_0} U_{N,0}\psi_N \| \leqslant C e^{c e^{c|t|}} N^{-1/8}.$$

- $\mathcal{U}_2(t)$ is an (approximate) Bogoliubov transformation and thus the approximate dynamics on Fock space just contains a phase and Bogoliubov transformations
- Crucial ingredient for the proof: unitary transformation that is cubic in annihilation/creation operators used to renormalize the generator on Fock space
- The initial condition allows $\psi_N = U_{N,0}^* e^{B_0} e^{\widetilde{B}} \Omega$ for $\widetilde{B} \approx \int \tau(x,y) a_x^* a_y^* - \text{h.c.}$, $\tau \in H^2$ and such states should be good approximations of the ground state in a trap
- The unitary $U_{N,t}$ is slightly modified compared to above
- *Application:* We show a central limit theorem for bounded one-particle observables