CRC TRR 352

# Return Probabilities for Simple Random Walks on Galton-Watson Trees


Markus Heydenreich[1] Peter Müller[2] Sara Terveer[2]

[1]University of Augsburg [2]Ludwig-Maximilians-Universität Munich


## Galton-Watson trees

Let $\mu$ be a probability distribution supported on $\mathbb{N}_0$ and $Z$ be a random variable distributed according to $\mu$. Let $m = \mathbb{E}Z$. The *Galton-Watson tree* (GWT) $\mathcal{T}$ with *offspring distribution* $\mu$ is a random rooted tree such that the number of children of each vertex is distributed as $Z$ and independent of the number of children of every other vertex.

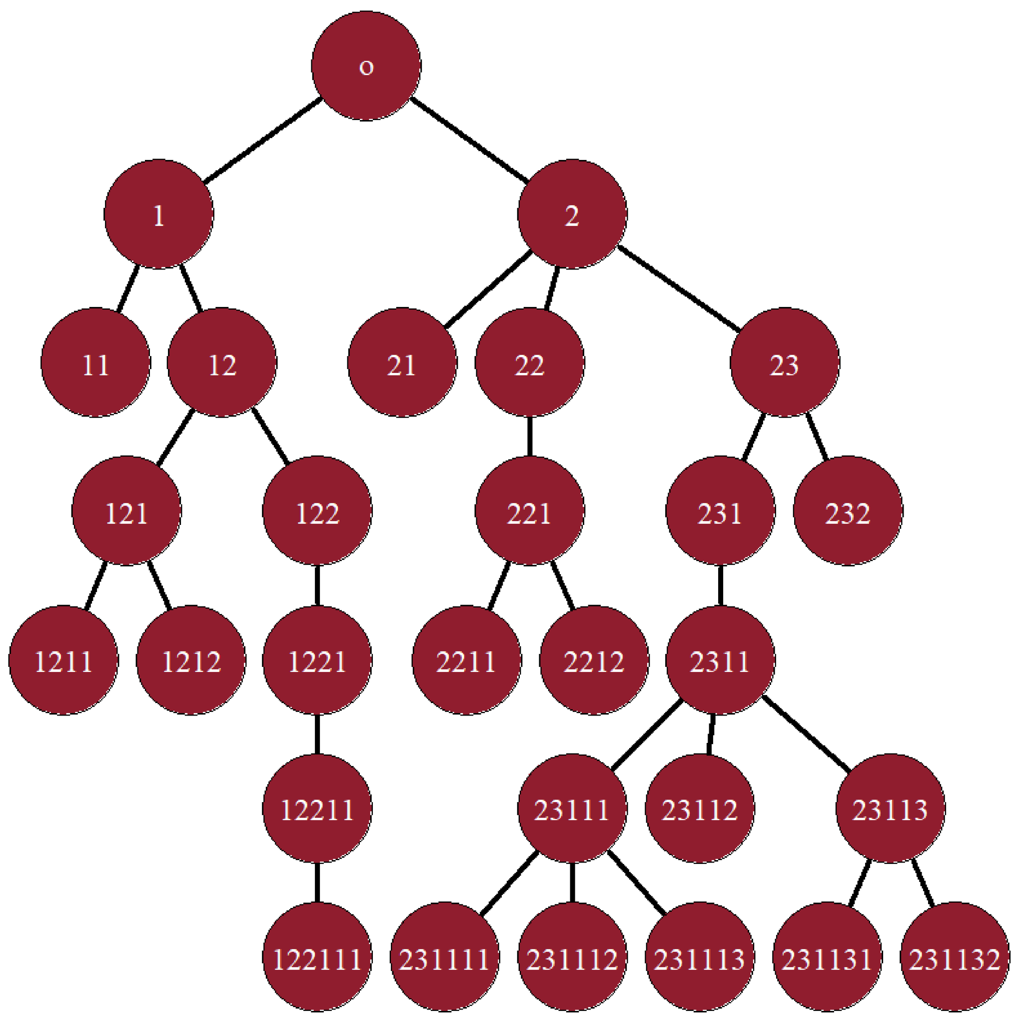


**Figure 1.** Realization the first six generations of a GWT with $\mathrm{Poi}(1.4)$ offspring distribution in Ulam-Harris labeling

- The probability measure $\mathbb{P}^*$ associated with the GWT is canonically determined by the offspring distribution.
- For every vertex $x \in \mathcal{T}$, the (sub-)tree consisting of $x$ and all its descendants is a GWT.
- The probability of extinction of a GWT is $q := \mathbb{P}^*(|\mathcal{T}| < \infty)$ and satisfies $q = 1$ iff $m \leq 1$.
- The probability measure conditioned on non-extinction is denoted by $\mathbb{P}(\cdot) = \mathbb{P}^*(\cdot \mid |\mathcal{T}| = \infty)$, the expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.

## Simple random walks

For a graph $G = (V, E)$, the simple random walk is a $V$-valued stochastic process $(X_t)_{t\in\mathbb{N}_0}$:

1. Choose initial vertex $v \in V$: $X_0 = v$.
2. For every $t \in \mathbb{N}_0$, choose the next vertex uniformly among the neighbors of the current one:

$$P^G(X_{t+1} = w \mid X_t = v) = \begin{cases} \frac{1}{\deg_G(v)}, & v \sim w \\ 0, & v \nsim w \end{cases}, \qquad v, w \in V$$

We abbreviate

$$P_v^G(\cdot) := P^G(\cdot \mid X_0 = v)$$

for the probability conditional on a fixed initial vertex $v \in V$.

If the graph $G$ is a rooted tree $T$ with root $o$, the *return probability* to the root in $t$ steps is $P_o^T(X_t = o)$.

## Annealed return probability

Combining the two previous concepts, we obtain random walks on Galton-Watson trees.

### Definition

The *annealed return probability* on a Galton-Watson tree is defined as

$$R_t := \mathbb{E}[P_o^{\mathcal{T}}(X_t = o)],$$

i.e. the expected return probability after $t$ steps with respect to the Galton-Watson tree conditioned on non-extinction.

Previously known results by Piau [3] give the following bounds for $R_t$:

| | | |
|---|---|---|
| (a) $\mu(0) = \mu(1) = 0$: | $\exp(-c't) \leq$ | $R_t \leq \exp(-ct)$ |
| (b) $\mu(0) = 0 \vee \mu(1) = 0$: | $\exp(-c't^{\frac{1}{3}}) \leq$ | $R_t$ |
| (c) $\mu(0) = 0$: | | $R_t \leq \exp(-ct^{\frac{1}{3}})$ |
| (d) $(\mu(j))_{j\in\mathbb{N}_0}$ finitely supported: | | $R_t \leq \exp(-ct^{\frac{1}{5}})$ |
| (e) general $(\mu(j))_{j\in\mathbb{N}_0}$: | | $R_t \leq \exp(-ct^{\frac{1}{6}})$ |

### Conjecture

Assume that the offspring distribution has exponentially decaying tails. Then there is a constant $c > 0$ such that for all $t \in \mathbb{N}_0$

$$R_t \leq \exp\left(-ct^{\frac{1}{3}}\right).$$

## Virág's approach

Virág [4] uses a variant of the isoperimetric inequality, the *anchored expansion* to bound the heat kernel decay for deterministic graphs. By adjusting parts of his approach with probabilistic methods, his result can be generalized to obtain upper bounds for $R_t$ (proof: see [2]):

### Theorem

1. For finitely supported offspring distributions, $R_t \leq \exp(-ct^{\frac{1}{3}})$.
2. For fast decaying offspring distributions, i.e. $\mu(j) \leq c_1 \cdot \exp(c_2 j^k)$ for every $j \in \mathbb{N}_0$, where $c_1, c_2 > 0$ and $k > 8$ are constants. Then there is a constant $c > 0$ such that for all $t \in \mathbb{N}_0$

$$R_t \leq \exp\left(-ct^{\frac{1}{3}-\frac{8}{3k}}\right).$$

## Watershed process

Aforementioned probabilistic methods require control over the typical structure of the graph. In particular, we require that the event of visiting a vertex with "many" children has low probability.

To minimize the number of vertices that need to be controlled, we perform construction and exploration of the tree simultaneously by generating offspring of an individual as it is first visited by the walk. For non-visited individuals, no progeny is generated and the vertex remains unmarked. We use an adaptation of the process introduced in [1].

### Definition

The *watershed process* is the process consisting of the constructed tree and the visited vertex under the law canonically determined by the offspring distribution and the transitions of the random walk.

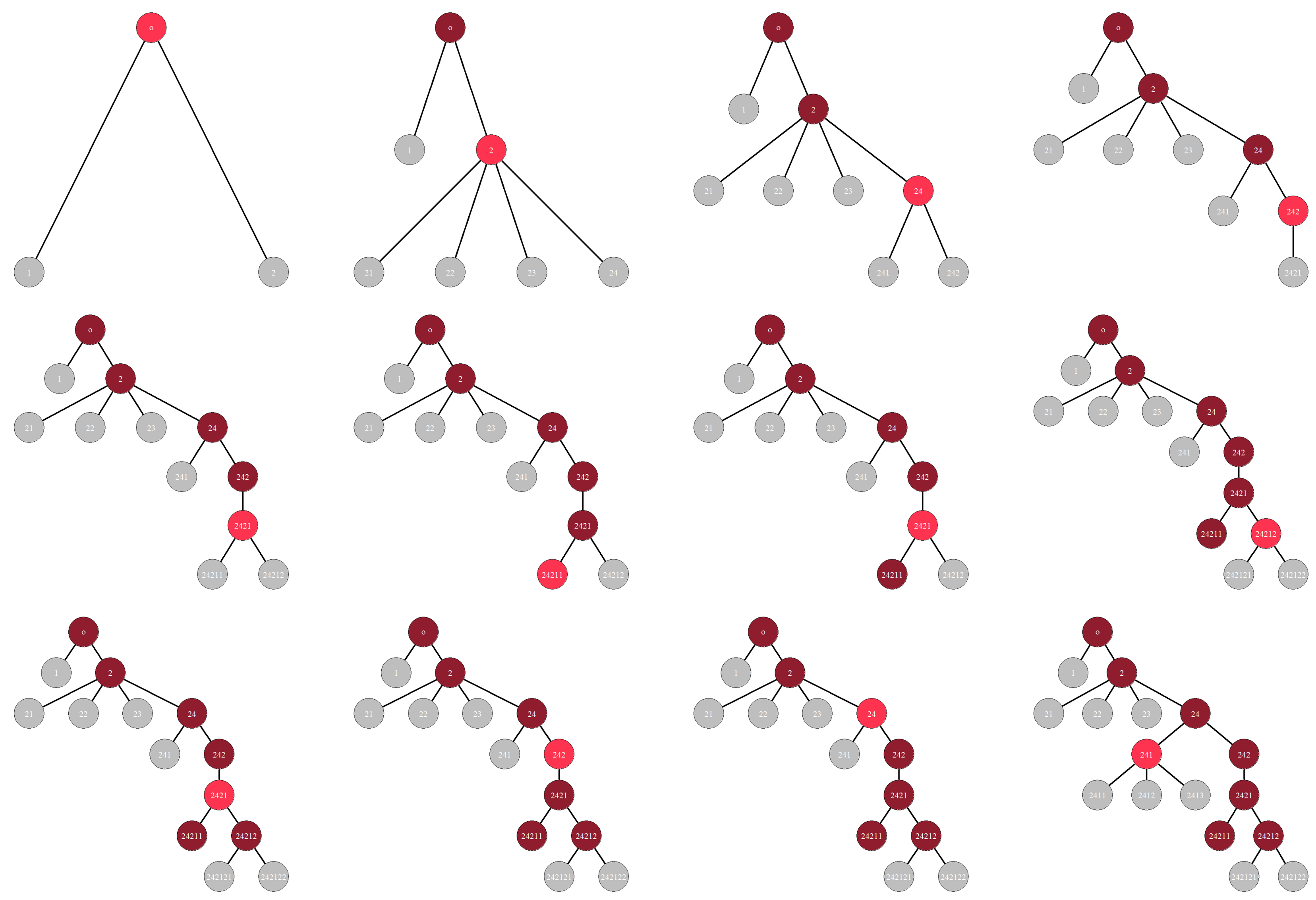

**Figure 2.** Left to right, top to bottom: Realization of the first 11 steps of the watershed process with offspring distribution $\mathrm{Poi}(1.4)$. Instead of having to generate the first 11 generations of the tree, we only generate the immediate offspring of visited individuals.

Instead of realizing a GWT first and then running $t$ steps of a random walk on it, therefore having to control the first $t$ generations of the tree, the watershed process allows to only control the number of children for at most $t$ vertices - thus improving the tail decay condition for the offspring distribution in the above theorem.

### Lemma

The watershed process has the same law as the annealed measure $\mathbb{E}[P_o^{\mathcal{T}}(\cdot)]$ of the random walk and the tree restricted to the vertices adjacent to the trace of the walk.

- conditional on a realization of the tree constructed by the watershed, the walk of the watershed is distributed as a simple random walk on the constructed tree,
- under the watershed law, the constructed tree "completed" by filling up non-visited vertices according to the offspring distribution has the same distribution as a GWT.

## References


[1] Alexander Drewitz, Gioele Gallo, and Alexis Prévost.
Generating Galton–Watson trees using random walks and percolation for the Gaussian free field.
*The Annals of Applied Probability*, 34(3):2844 – 2884, 2024.

[2] Peter Müller and Jakob Stern.
On the return probability of the simple random walk on Galton–Watson trees.
arXiv:2402.01600, 2024.

[3] Didier Piau.
Théorème central limite fonctionnel pour une marche au hasard en environnement aléatoire.
*The Annals of Probability*, 26(3):1016 – 1040, 1998.

[4] Bálint Virág.
Anchored expansion and random walk.
*Geometric & Functional Analysis*, 10(6):1588 – 1605, 2000.