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Two results about spectral gaps of quantum spin systems on general graphs

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Setting: frustration-free quantum spin systems

We consider quantum spin systems defined on a graph Γ.
Hilbert space: For $\Lambda \subset \Gamma$ finite, set $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^d$
Hamiltonian:

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi(X)$$

with Φ a positive semidefinite, finite-range interaction.

Assumption: H_Λ is frustration-free, i.e., $\inf \operatorname{spec} H_\Lambda = 0$.
Spectral gap: $\gamma_\Lambda = \inf(\operatorname{spec} H_\Lambda \setminus \{0\}) > 0$.

Project A8:

- work package A: extensions of finite-size criteria
- work package B: applications of finite-size criteria

Result A on critical gap thresholds (WP A)

Suppose the graph Γ can be embedded in $\mathbb{R}^D$. For each $k \in \mathbb{N}$, introduce the Euclidean rectangle

$$R(k) = [l_{k+1}, \dots, l_{k+D}], \qquad l_k = \left(\frac{3}{2}\right)^{k/D}$$

and let $\mathcal{F}_k$ = subsets of Γ contained in $R(k)$ up to translations and permutations of the coordinates.

Theorem (L–Lucia 2024 [7]) *Set* $\gamma_{\mathcal{F}_k} = \inf_{\Lambda\in\mathcal{F}_k} \gamma_\Lambda$. *Suppose that*

$$\inf_{k\geq 1} \gamma_{\mathcal{F}_k} = 0.$$

Then, for every $\epsilon > 0$,

$$\gamma_{\mathcal{F}_k} = o\left(\frac{k^{4+\epsilon}}{l_k^2}\right), \qquad k \to \infty$$

Comments:

1. Ignoring $\log$'s, this says that if the gap closes in the limit, then it must close at least like the inverse-square l_k^{-2}
2. result proves inverse-square critical gap scaling is universal property of frustration-free Hamiltonians. Prior results for nearest-neighbor interactions on specific graphs [1, 3, 8] or assumptions on the g.s. [9].
3. CFT gap scaling would be associated with $\gamma_{\mathcal{F}_k} \sim l_k^{-1}$.
$\to$ informal no-go result: finite-range frustration-free Hamiltonians cannot produce CFT's in continuum limit.

Tool A.1: divide-and-conquer approach

Kastoryona–Lucia [6] developed an iterative scheme to relate overlaps of g.s. on different system sizes

$$\inf_{k\geq 1} \gamma_{\mathcal{F}_k} \geq \gamma_{\mathcal{F}_{k_0}} \prod_{k=k_0}^{\infty} \frac{1-\delta_k}{1+k^{-1-\epsilon}}$$

and $\delta_k = \sup_{(A,B)} \|P_A P_B - P_{A\cup B}\|$ for $A, B \in \mathcal{F}_k$ s.t. $d(A\setminus B, B\setminus A) \sim l_k$.

Tool A.2: refined Detectability Lemma

The divide-and-conquer approach gives lower bound on gap via overlaps. For bootstrapping, we need a converse.

Lemma (refined overlap bound): $\delta_k \leq C_1 \exp\left(-C_2 \sqrt{\gamma_{\mathcal{F}_k}} \frac{l_k}{k^{1+\epsilon}}\right)$.

This improves previous bound by [6] from $\gamma_{\mathcal{F}_k}$ to $\sqrt{\gamma_{\mathcal{F}_k}}$ and its why we get inverse-square scaling.

Proof idea: generalized refined Detectability Lemma à la Gosset-Huang [2] (coarse-graining and then Chebyshev polynomials)

Result B on gaps of random Hamiltonians (WP B)

Let G be a $d\times d$ GOE matrix and set

$$v = \sum_{i,j=1}^{d} \frac{G_{i,j}}{\sqrt{\operatorname{Tr}(G^2)}} e_i \otimes e_j \in \mathbb{C}^d \otimes \mathbb{C}^d$$

Given a finite graph Λ, consider the random Hamiltonian

$$H_\Lambda = \sum_{x\sim y} Q_{xy}, \qquad Q_{xy} = |v\rangle\langle v| \otimes 1_{\Lambda\setminus\{x,y\}}. \tag{1}$$

i.e., n.n. interaction = a fixed random rank-1 projector.

Comment: symmetry of GOE matrix C implies that Q_{xy} acts on undirected edges (for simplicity only)

Theorem (Hunter-Jones–L 2024 [4]) *Assume that* $d, k \geq 1$ *satisfy*

$$\frac{d}{2(2e^2 \log d + 1/4)^2} > 2k - 2. \tag{2}$$

Then, there ex. $c > 0$ *and an event* Ω *with probability at least*

$$\mathbb{P}(\Omega) \geq 1 - 2e^{-d^2/16} \tag{3}$$

such that for every $\omega \in \Omega$ *the following holds:*
For every finite graph Λ *of maximal degree* k, $\gamma_\Lambda \geq c > 0$.

Comments:

1. This gives many gapped Hamiltonians. The event Ω and the lower bound c are universal (only depend on k and not otherwise on Λ)
2. E.g., if $k = 2$, then $d \geq 15$ ensures a gap with 99% probability. (Current multi-mode cavity experiments can reach $d \approx 10$)
3. The Hamiltonians (1) are automatically frustration-free (for suff. large d). Proved by a cluster expansion and QSAT crit. [10, 5]

Tool B.1: three-vertex criterion à la Knabe

Proposition (three-vertex criterion)
Let $H_3 = Q_{1,2} + Q_{2,3}$. *For every finite graph* Λ *of maximal degree* k,

$$\gamma_\Lambda \geq 2(k-1)\left(\gamma_3 - \frac{2k-3}{2k-2}\right). \tag{4}$$

Proof idea: Squaring the Hamiltonian and combinatorics.
Corollary: It suffices to prove $\gamma_3 > \frac{2k-3}{2k-2}$.

Tool B.2: concentration bounds

By Fannes-Nachtergaele-Werner lemma and explicit calculations, it suffices to control

$$\|Q_{1,2} Q_{2,3}\| = \frac{\|G\|^2}{\operatorname{Tr}(G^2)}$$

By Lévy concentration, suffices to control $\mathbb{E}[\|G\|^2] \leq (\mathbb{E}[\operatorname{Tr}(G^{2p})])^{1/p}$.
By comparing with GUE it is easy to prove

$$\left(\mathbb{E}[\operatorname{Tr}(G^{2p})]\right)^{1/p} \leq \left(\frac{(2p)!}{2^p p!} d\right)^{1/p}$$

The optimal choice is essentially $p = \log d$ which gives $2e^2 \log d$ in (2).

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