The Spectral Gap and Low-Energy Spectrum in Mean-Field Quantum Spin Systems

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Mean-field Quantum Spin Systems

Total spin-vector of N-qubits: $\mathbf{S} = \sum_{n=1}^{N} \mathbf{S}(n)$ on $\mathcal{H}_N = \bigotimes_{n=1}^{N} \mathbb{C}^2$ with $\mathbf{S}(n) = \mathbb{1} \otimes \cdots \otimes \mathbf{s} \otimes \cdots \otimes \mathbb{1}$

and spin vectors $\mathbf{s} = (s_x, s_y, s_y)$ consisting of the three generators of $SU(2)$:

$$s_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Mean-field Hamiltonian = Weyl ordered, self-adjoint polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ of the total spin:

$$H = N\,\mathrm{P}\left(\tfrac{2}{N}\mathbf{S}\right)$$

Examples: Lipkin-Meshkov-Glick model $P(\mathbf{m}) = -\alpha m_y^2 - \beta m_z^2 - \gamma m_x$ with $\alpha, \beta, \gamma \in \mathbb{R}$.

Special case: $\alpha = 0$, $\beta = 1$ Quantum Curie-Weiss model

Block diagonalization: $\mathcal{H}_N \equiv \bigoplus_{J=\frac{N}{2}-\lfloor\frac{N}{2}\rfloor}^{N/2} \bigoplus_{\alpha=1}^{M_{N,J}} \mathbb{C}^{2J+1}$, $\quad M_{N,J} = \frac{2J+1}{N+1}\binom{N+1}{\frac{N}{2}+J+1}$.

$$H = \bigoplus_{J=\frac{N}{2}-\lfloor\frac{N}{2}\rfloor}^{N/2} \bigoplus_{\alpha=1}^{M_{N,J}} H_{J,\alpha}$$

Methods I: Semiclassical Description

Spin J operators: $[S_x, S_y] = iS_z$ (and cyclically) $\quad S_\pm = S_x \pm iS_y \quad$ on Hilbert space $\mathbb{C}^{2J+1}$

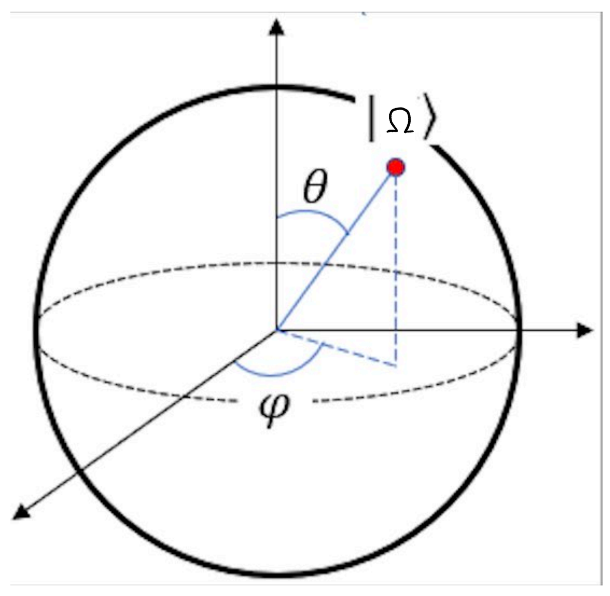

Figure 1. The Bloch ball B_1 and sphere S^2

Bloch coherent state: $\Omega = (\theta, \varphi), \quad 0 \leq \theta \leq \pi, \ 0 \leq \varphi \leq 2\pi$

$$|\Omega\rangle := \exp\left(\frac{\theta}{2}\left(e^{i\varphi}S_- - e^{-i\varphi}S_+\right)\right)|J\rangle$$

Symbols of a linear operator G on $\mathbb{C}^{2J+1}$:

Lower: $\quad g(\Omega) := \langle\Omega|G|\Omega\rangle \qquad$ Upper: $\quad G = \frac{2J+1}{4\pi}\int d\Omega\, G(\Omega)\,|\Omega\rangle\langle\Omega|$

Block (J, α)-Hamiltonians $H_{J,\alpha} = N\,P\left(\frac{2}{N}\mathbf{S}\right)$ with spin-J operator $\mathbf{S} = (S_x, S_y, S_z)^T$ on $\mathbb{C}^{2J+1}$ have the consistent *semiclassical symbol* $P : B_1 \to \mathbb{R}$ thanks to:

Quantitative Duffield Theorem [2]: For some $C \in [0, \infty)$:

Lower. $\quad \sup_N \sup_{0 \leq J \leq N/2} \sup_\Omega \left|\langle\Omega, J|H_{J,\alpha}|\Omega, J\rangle - N\,P\left(\frac{2J}{N}\,\mathbf{e}(\Omega)\right)\right| \leq C.$

Upper. $\quad \sup_N \sup_{J,\alpha} \left\| H_{J,\alpha} - \frac{2J+1}{4\pi}\int d\Omega\, N\,P\left(\frac{2J}{N}\,\mathbf{e}(\Omega)\right)|\Omega, J\rangle\langle\Omega, J| \right\| \leq C,$

Immediate implication given [1, 4, 7] is the **semiclassics for the free energy:**

$$\lim_{N\to\infty} N^{-1}\ln\operatorname{tr}\exp\left(-\beta N P\left(\tfrac{2}{N}\mathbf{S}\right)\right) = \max_{r\in[0,1]}\left\{I(r) - \beta\min_{\Omega\in S^2} P\left(r\mathbf{e}(\Omega)\right)\right\}.$$

with the binary entropy $I(r) = -\frac{1+r}{2}\ln\frac{1+r}{2} - \frac{1-r}{2}\ln\frac{1-r}{2}$.

Methods II: Phase Space Geometry at Mininma

Recipe from Fluctuation Theory: Quadratic approximations at the **minima** $\mathbf{m}_0$ $(\mathbf{m}_1, \dots, \mathbf{m}_L) \in B_1$ of P determine the low-energy spectra and spectral gap of H.

See e.g. [6, 3] *for implementations.*

- Simple-minded Taylor approximation:

$$P(\mathbf{m}) = P(\mathbf{m}_0) + \nabla P(\mathbf{m}_0)\cdot(\mathbf{m} - \mathbf{m}_0) + \frac{1}{2}(\mathbf{m} - \mathbf{m}_0)D_P(\mathbf{m}_0)(\mathbf{m} - \mathbf{m}_0) + \mathcal{O}\left((\mathbf{m} - \mathbf{m}_0)^3\right)$$

- Case $|\mathbf{m}_0| = 1$: $\quad \nabla P(\mathbf{m}_0) = -|\nabla P(\mathbf{m}_0)|\mathbf{e}_{\mathbf{m}_0}$ does not vanish in general!
- Fluctuations for fixed J occur only in the angular directions:

Local chart Φ : ran $Q_\perp \to T_{\mathbf{m}_0}S^2$ and its quadratic approximation to $P \circ \Phi$:

$$D_P^\perp(\mathbf{m}_0) := Q_\perp D_P(\mathbf{m}_0)Q_\perp + |\nabla P(\mathbf{m}_0)|Q_\perp, \qquad Q_\perp := \mathbb{1}_{\mathbb{R}^3} - \mathbf{e}_{\mathbf{m}_0}^T\mathbf{e}_{\mathbf{m}_0}.$$

Low Energy Spectra & Gaps

Theorem [5]: In case P has a **unique global minimum** at $\mathbf{m}_0 \in S^2$ at which $|\nabla P(\mathbf{m}_0)|, \det D_P^\perp(\mathbf{m}_0) > 0$, the lowest eigenvalues of H coincides with points in the set

$$NP(\mathbf{m}_0) + (2k-1)|\nabla P(\mathbf{m}_0)| + (2m+1)\sqrt{\det D_P^\perp(\mathbf{m}_0)} + o(1)$$

where $m \in \mathbb{N}_0$ and $k = N/2 - J \in \mathbb{N}_0$ relates to the total spin J. In particular, the ground-state is unique and found at $J = N/2$, and the **spectral gap** is

$$\operatorname{gap} H = 2\min\left\{|\nabla P(\mathbf{m}_0)|, \sqrt{\det D_P^\perp(\mathbf{m}_0)}\right\} + o(1).$$

Theorem [5]: In case P has a **finite number of global minima** at $\mathbf{m}_1, \dots, \mathbf{m}_L \in S^2$ at which $|\nabla P(\mathbf{m}_l)|, \det D_P^\perp(\mathbf{m}_l) > 0$, the lowest eigenvalues of H coincides with points in the set

$$NP(\mathbf{m}_l) + (2k-1)|\nabla P(\mathbf{m}_l)| + (2m+1)\sqrt{\det D_P^\perp(\mathbf{m}_l)} + o(1)$$

where $l \in \{1, \dots, L\}$, $m \in \mathbb{N}_0$ and $k = N/2 - J \in \mathbb{N}_0$ relates to the total spin J.

Quantum Curie-Weiss model: $\quad P(\mathbf{m}) = -m_x^2 - \gamma m_z$

Paramagnetic phase $\gamma > 2$: $\quad \mathbf{m}_0 = (0, 0, 1)^T \qquad \operatorname{gap} H = 2\sqrt{\gamma(\gamma - 2)}$

Ferromagnetic phase $0 \leq \gamma < 2$: $\quad \mathbf{m}_0^\pm = (\pm\sqrt{1 - \gamma^2/4}, 0, \gamma/2)^T$

$$|\nabla P(\mathbf{m}_0^\pm)| = 2, \quad \det D_P^\perp(\mathbf{m}_0^\pm) = 4 - \gamma^2$$

Nearly doubly degenerate ground states separated by a spectral gap $4\sqrt{1 - \gamma^2/4}$ from the rest of the spectrum.

See also [8, 9] *elated results and further motivations*

Theorem [5]: In case the **unique global minimum** $\mathbf{m}_0 \in B_1$ is at $0 < |\mathbf{m}_0| < 1$ with $D_P(\mathbf{m}_0) > 0$. Then the ground state is contained in a subspace with total spin J with $|J - N|\mathbf{m}_0|/2| \leq \mathcal{O}(\sqrt{N})$ and

$$E_0(H) = E_0(H_{J,\alpha}) = NP(\mathbf{m}_0) + |\mathbf{m}_0|\sqrt{\det D_P^\perp(\mathbf{m}_0)} + o(1).$$

For any J with $|J - N|\mathbf{m}_0|/2| \leq o(\sqrt{N})$ the ground-state energy $E_0(H_{J,\alpha})$ is still given by the above formula.

Methods III: Controlling Fluctuations

Overall strategy: Investigate spectra of each block $H_{J,\alpha}$ separately, and in each block apply:

Krein-Feshbach-Schur method: [10] Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, and E and F orthogonal projections with $E + F = \mathbb{1}_\mathcal{H}$. Assume that $a < \inf\operatorname{spec} FAF$ and let $R(a) = (FAF - aF)^{-1}$ stand for the block inverse on $F\mathcal{H}$. Then

$$a \in \operatorname{spec} A \quad \text{if and only if} \quad 0 \in \operatorname{spec} EAE - aE - EAFR(a)FAE.$$

In particular, the eigenvalues $\alpha_0(A) \leq \alpha_1(A) \leq \dots$ of A (counted with multiplicities) and the respective eigenvalues of EAE satisfy

$$|\alpha_j(A) - \alpha_j(EAE)| \leq \frac{\|EAF\|^2}{\operatorname{dist}(\operatorname{spec} FAF, a)}$$

provided $\alpha_j(A) < a < \operatorname{spec} FAF$.

As illustration, suppose there is a **unique minimizers** at $\mathbf{m}_0 = \mathbf{e}_z \in S^2$.

Subspace of $\mathbb{C}^{2J+1}$ at minimizing direction is spanned by z-basis:

$$\mathcal{H}_J^K = \operatorname{span}\left\{|N/2 - k\rangle \in \mathbb{C}^{2J+1}\,\middle|\, k \in \{0, , 1, \dots, K\}\right\}$$

- $\left\|(2S_z/N - 1)P_J^K\right\| \leq K/N.$
- Size of **fluctuation operators** $\xi \in \{x, y\}$: $\quad \left\|\sqrt{\tfrac{2}{N}}S_\xi\, P_J^K\right\| \leq C_J\sqrt{K}.$
- $\left[\sqrt{\tfrac{2}{N}}S_x, \sqrt{\tfrac{2}{N}}S_y\right] = i\,2S_z/N = i\,(1 + o(1))$ $\quad$ asymptotically **hamonic oscillator** algebra.

Quadratic approximation of P at $\mathbf{m}_0$:

$$Q_J(\mathbf{m}_0) := N\,P(\mathbf{m}_0) + 2\left(\mathbf{S} - \tfrac{N}{2}\mathbf{m}_0\right)\cdot\nabla P(\mathbf{m}_0) + \tfrac{2}{N}\mathbf{S}\cdot D_P^\perp(\mathbf{m}_0)\mathbf{S} \quad \text{on } \mathbb{C}^{2J+1}.$$

$$\left\|(H_{J,\alpha} - Q_J(\mathbf{m}_0))P_J^{K_N}\right\| = o(1) \qquad \text{as long as } K_N = o(N^{1/3}) \text{ and } J \geq N/2 - K_N.$$

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