CRC TRR 352

Charge Gap $\geq$ Neutral Gap

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Dipol-, Charge- and Translation Symmetry

We consider spinless Fermions or Bosons on a ring $[1, L] := \mathbb{Z}/L\mathbb{Z}$. The fermionic or bosonic creation $(a_j^\dagger)$ and annihilation (a_j) operators corresponding to the canonical basis in the one-particle Hilbert space $\ell^2([1, L])$ are defined in the fermionic or bosonic Fock space $\mathcal{F}$.

Symmetries implemented by unitaries on $\mathcal{F}$:

Translation	$T^\dagger a_j T = a_{(j-1) \mod L}$	for all $j \in [1, L]$;
Charge	$U = \exp\left(\frac{2\pi i}{L} N\right)$	generated by the particle-number operator $N := \sum_{j=1}^{L} a_j^\dagger a_j$;
Dipol	$V = \exp\left(\frac{2\pi i}{L} D\right)$	generated by the dipole operator $D := \sum_{j=1}^{L} j\, a_j^\dagger a_j$.

Relations: $VT = UTV, \quad UT = TU, \quad UV = VU.$

General Set-Up: Assumptions

The **Hamiltonian** is self-adjoint on $\mathcal{F}$ and of the form $H = \sum_{m_0 \leq m \leq m_1} \sum_{\substack{j_1 \dots j_m \\ k_1 \dots k_m}} W_{j_1 \dots j_m}^{k_1 \dots k_m}, a_{j_1}^\dagger \dots a_{j_m}^\dagger a_{k_m} \dots a_{k_1}$

with m-body coefficients $W_{j_1 \dots j_m}^{k_1 \dots k_m} \in \mathbb{C}$ with $m_0, m_1 \in \mathbb{N}$ such that:

Symmetries: H commutes with T, U, V.

Consequence of charge conservation: $H = \bigoplus_{n \geq 0} H_n$ with n-particle Hamiltonians $H_n \geq 0$.

We assume the existence of a particle number $n_p \in \mathbb{N}$ corresponding to some **fractional filling**

$$\frac{q}{p} = \frac{n_p}{L}$$

with $1 \leq q < p$ coprime, at which:

Positivity: $H_{n_p} \geq 0$

Ground States: $\ker H_{n_p} = \operatorname{span}\{\varphi, T\varphi, \dots, T^{p-1}\varphi\}$ with some normalized $\varphi \in \ker H_{n_p}$ such that:

- φ is an eigenvector of U and V.
- φ is p-periodic: $T^p \varphi = \varphi$.

Orthonormality: The vectors $T^j \varphi$ with $j = 0, \dots, p-1$ form an orthonormal basis of $\ker H_{n_p}$, since they are eigenstates of V with distinct eigenvalues $\exp\left(2\pi i \left(\frac{d}{L} + \frac{j}{p}\right)\right)$, with some $d \in \mathbb{Z}$.

(Maximal) filling pattern: $\max_j \langle \varphi, N_j \varphi \rangle \leq \sum_{j=1}^{p} \langle \varphi, N_j \varphi \rangle = q$ with $N_j = a_j^\dagger a_j$.

Motivation & Examples

Fractional Quantum Hall systems (FQHS) with inverse filling $p \in \{2, 3, \dots\}$ in torus geometry. Fermionic if p is odd and bosonic if p is even.

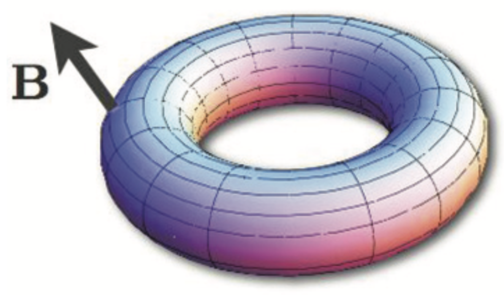

Figure 1. Torus geometry $[1, L] = \mathbb{Z}/L\mathbb{Z}$

1. **Haldane pseudopotentials** [1] $H = \sum_s B_s^* B_s$, with $B_s := \sum_k{}' F_p(k)\, a_{s+k} a_{s-k}$.

   E.g. $p = 2$ bosonic model $F_2(k) = \sum_{j \in \mathbb{Z}} e^{-\alpha^2 (k + jN)^2}$,

   or $p = 3$ fermionic model $F_3(k) = \sum_{j \in \mathbb{Z}} \alpha(k + jN) e^{-\alpha^2 (k + jN)^2}$.

2. **truncated Haldane pseudopotentials** [2, 3, 4], e.g., $p = 3$ fermionic model with $\kappa \geq 0$ and $\lambda \in \mathbb{C}$:

$$H = \sum_{j=1}^{L} \left(N_j N_{j+2} + \kappa\, Q_j^\dagger Q_j \right), \quad \text{with} \quad Q_j := a_{j+1} a_{j+2} - \lambda\, a_j a_{j+3}.$$

Dipole-, charge- and translation symmetric with **maximal filling fraction** $\frac{1}{p} = \frac{n_p}{L}$.

- p-periodic **Laughlin wavefunction** φ and its translates $T^j \varphi$, $j \in \{1, \dots, p-1\}$, are ground-states at maximal filling.
- **Spectral gaps** neutral gap at maximal filling vs. charge gap at higher filling, see e.g. [5].

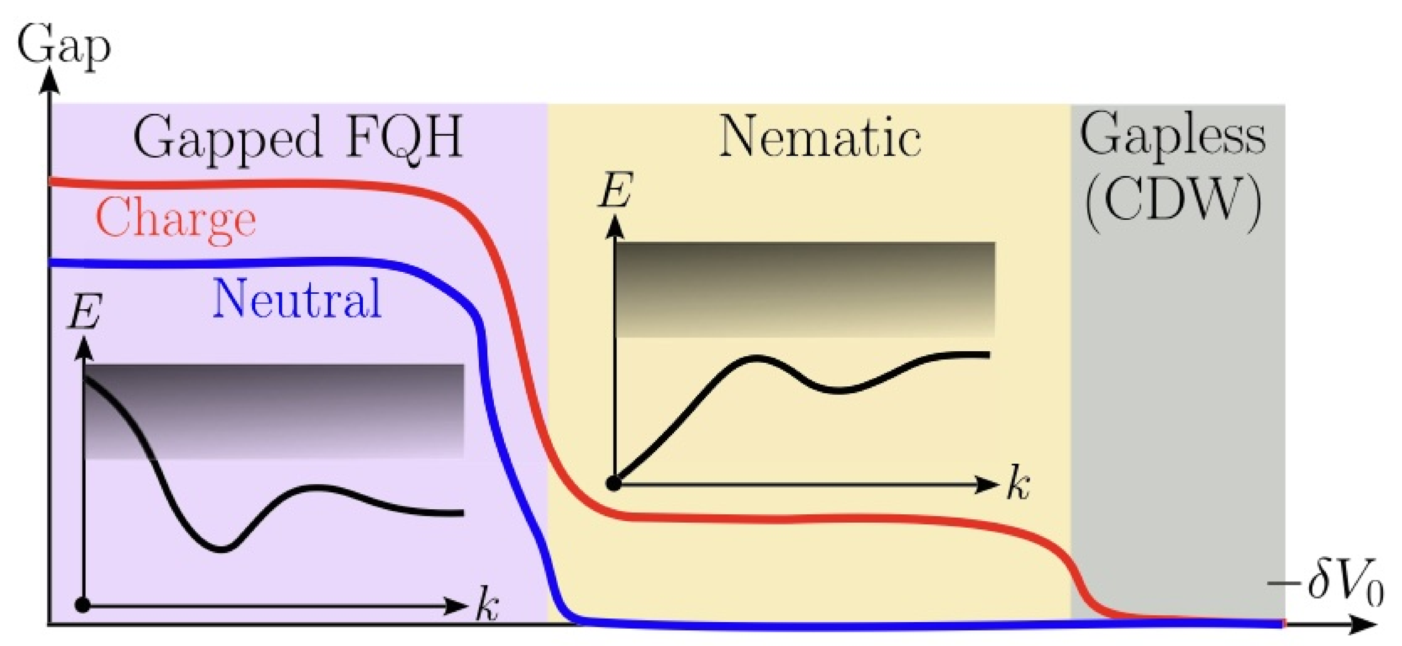

Figure 2. Sketch of the charge and neutral gap in a FQHS taken from [5]. The nematic phase is characterized by the vanishing of the neutral gap while the charge gap remains open.

Hallmark of incompressibility

By assumption: $\dim \ker H_{n_p} = p$.

Theorem: If $n > n_p$: $\dim \ker H_n = 0$ assuming $L \geq \frac{p^2}{q}$ for fermions and $L \geq (q+1)\frac{p^2}{q}$ for bosons.

Spectral gap at filling $(n = n_p)$ respectively **ground-state energy** $(n > n_p)$:

$$\operatorname{gap} H_n := \inf_{\substack{\psi \perp \ker H_n \\ \|\psi\| = 1}} \langle \psi, H_n \psi \rangle.$$

Neutral versus Charge Gap

- **Neutral (= Haldane) gap:** $\operatorname{gap} H_{n_p}$
- **Charge gap:** $\operatorname{gap} H_{n_p+1}$

Theorem: In case of p- commensurate filling, i.e. $L = mp^2/q$ for some $m \in \mathbb{N}$, for any $n \geq n_p$:

Fermionic case: $\operatorname{gap} H_{n+1} \geq \frac{n_p}{n_p + 1 - m_0} \operatorname{gap} H_{n_p}$.

Bosonic case: $\operatorname{gap} H_{n+1} \geq \frac{n_p - q}{n_p + 1 - m_0} \operatorname{gap} H_{n_p}$ (requiring $m_0 \geq q + 1$)

Methods I: Inductive Relation

For $H_n \geq 0$ with $n \geq m_1$:

$$H_{n+1} \geq \frac{1}{n + 1 - m_0} \sum_{j=1}^{L} a_j^\dagger\, H_n\, a_j \qquad \text{(IR)}$$

Follows from the 'combinatorial' identity for any $m \in \mathbb{N}$ and arbitrary $j_1, \dots, j_m, k_m, \dots, k_1 \in [0, L]$:

$$N a_{j_1}^\dagger \cdots a_{j_m}^\dagger a_{k_m} \cdots a_{k_1} = m a_{j_1}^\dagger \cdots a_{j_m}^\dagger a_{k_m} \cdot\ a_{k_1} + \sum_j a_j^\dagger \left[a_{j_1}^\dagger \cdots a_{j_m}^\dagger a_{k_m} \cdots a_{k_1} \right] a_j.$$

Methods II: Gap Comparison

For any Hamiltonian satisfying (IR) and $H_n \geq 0$ with $n \geq m_1$:

$$\operatorname{gap} H_{n+1} \geq \frac{\operatorname{gap} H_n}{n + 1 - m_0} \left(n + 1 - \|G^{(n)}\| \right).$$

where $\|G^{(n)}\|$ is the operator norm of the $q_n L \times q_n L$ matrix with entries

$$G_{\alpha j, \beta k}^{(n)} := \langle a_j^\dagger \varphi_\alpha, (1 - P_{n+1})\, a_k^\dagger \varphi_\beta \rangle, \qquad j, k \in \{1, \dots, L\},\ \alpha, \beta \in \{1, \dots, q_n\},$$

(φ_α) denotes an orthonormal basis of $\ker H_n$ of dimension $q_n := \dim \ker H_n$, and P_{n+1} is the orthogonal projection onto $\ker H_{n+1}$.

- Implies $\operatorname{gap} H_{n+1} \geq \frac{n+1}{n + 1 - m_0} \operatorname{gap} H_n$ for any $n > n_p$.
- Second-quantized, m-body analog of induction on particle-number for the proof [6] of the spectral gap in the Kac-Master equation.
- *Open challenge, e.g. for Haldane gap:* Estimate $\left\|G^{(n)}\right\|$!?

Methods III: Many-Body Gram Matrices

In case $n = n_p$, one has $P_{n_p+1} = 0$ and $G^{(n_p)} \equiv G$ is the $pL \times pL$ Gram matrix:

$$G_{jk, j'k'} = \langle a_j^\dagger T^{k-1} \varphi, a_{j'}^\dagger T^{k'-1} \varphi \rangle \quad \text{labeled by } j, j' \in \{1, \dots, L\},\ k, k' \in \{1, \dots, p\}.$$

Main idea: Orthogonality due to distinct eigenvalues of V:

$$\langle a_j^\dagger T^{k-1} \varphi, a_{j'}^\dagger T^{k'-1} \varphi \rangle = 0 \quad \text{unless} \quad (kn + j) \mod L = (k'n + j') \mod L,$$

Block diagonalization $G = \bigoplus_{\gamma=1}^{L} G(\gamma)$ with $p \times p$ blocks

$$G(\gamma) := \left[\left\langle a_{(\gamma - nk) \mod L}^\dagger T^{k-1} \varphi, a_{(\gamma - nl) \mod L}^\dagger T^{l-1} \varphi \right\rangle \right]_{k, l \in \{1, \dots, p\}}$$

Structure of blocks: $G(\gamma) = \mathbb{1} \pm F(\gamma)$ with $(-)$ for fermions and $(+)$ for bosons and

$$F(\gamma) := \left[\left\langle a_{(\gamma + nk) \mod L} T^{k-1+(k+l)n} \varphi, a_{(\gamma + nl) \mod L} T^{l-1+(k+l)n} \varphi \right\rangle \right]_{k, l \in \{1, \dots, p\}}$$

In case of p-commensurate filling, i.e. $n_p = 0 \mod p$, the p-periodicity of φ implies that $F(\gamma)$ is another Gram matrix. Hence $F(\gamma) \geq 0$ such that $0 \leq G(\gamma) \leq \mathbb{1}$ in the case of fermions.

In the case of bosons, the trace bound:

$$\operatorname{tr} F(\gamma) = \sum_{k=1}^{p} \left\langle \varphi, N_{(\gamma + 1 - (n+1)k) \mod p}\, \varphi \right\rangle = \sum_{k=1}^{p} \langle \varphi, N_k\, \varphi \rangle = q,$$

implies $0 \leq F(\gamma) \leq q\, \mathbb{1}$ and hence $0 \leq G(\gamma) \leq (1 + q)\, \mathbb{1}$.

Remark: The Gram blocks $G(\gamma)$ should generally be diagonally dominated!

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