Spectral inequality for the Landau Hamiltonian

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(joint work with Matthias Täufer)

$H_B = \left(-i\nabla + \frac{B}{2}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}\right)^2$

Landau Operator

Magnetic derivatives with magnetic field strength $B \in \mathbb{R}^+$

$$\begin{pmatrix}\tilde\partial_1\\ \tilde\partial_2\end{pmatrix} = \begin{pmatrix} i\partial_1 - \frac{B}{2}x_2\\ i\partial_2 + \frac{B}{2}x_1\end{pmatrix}.$$

Landau Operator

$$H_B = \tilde\partial_1^2 + \tilde\partial_2^2 \quad \text{self-adjoint in } L^2(\mathbb{R}^2).$$

Spectrum consists of eigenvalues at **Landau Levels**

$$\sigma(H_B) = \{B, 3B, 5B, \dots\}.$$

Models an electron in a plane, subject to a perpendicular magnetic field.

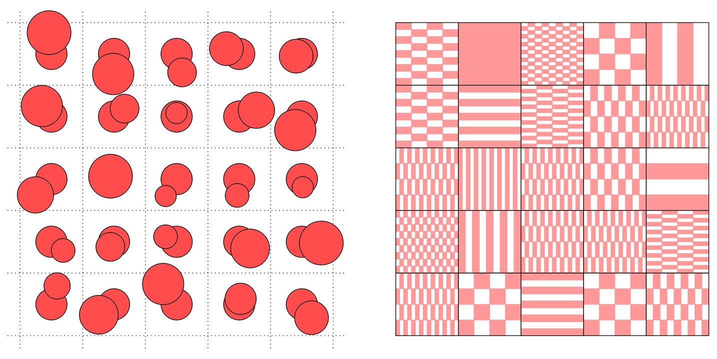
Thick sets

$S \subset \mathbb{R}^2$ is (ℓ,ρ)-**thick** if it is

(i) measurable,

(ii) $|[x_1+\ell) \times [x_2+\ell)| \geq \rho\ell^2$ for all $\binom{x_1}{x_2} \in \mathbb{R}^2$.

Thick sets can be very rough.



$\tilde\partial_1(u\overline{v})(x) = \overline{v}(x)\tilde\partial_1 u(x) - u(x)\overline{\tilde\partial_1 v}(x)$

$\operatorname{supp}\hat f \subset D_r \;\Rightarrow\; \|f\|^2_{L^2(\mathbb{R}^2)} \leq \exp\left(C(1+\ell r)\right)\|f\|^2_{L^2(S)}$

Theorem

For all $B \geq 0$, all **thick** $S \subset \mathbb{R}^2$, all $E \geq B$, and all $f \in \operatorname{Ran}\chi_{(-\infty,E]}(H_B)$

$$\|f\|^2_{L^2(\mathbb{R}^2)} \leq \exp\left(C(1+\ell\sqrt{E}+\ell^2 B)\right)\|f\|^2_{L^2(S)}, \qquad (1)$$

where $C \sim -\ln\rho > 0$.

$[\tilde\partial_1, \tilde\partial_2] = -iB$

$B\sum_{k=0}^{\infty}\frac{1}{2^k}\exp\left(-\frac{B}{4}\left(x_1^2+(x_2-2^{2^k})^2 - 2ix_1 2^{2^k}\right)\right)$

Strategy of proof

Magnetic Bernstein inequality For all $f \in \operatorname{Ran}\chi_{(-\infty,E]}(H_B)$, $m \geq 1$,

$$\sum_{\alpha\in\{1,2\}^m}\|\tilde\partial_{\alpha_1}\tilde\partial_{\alpha_2}\dots\tilde\partial_{\alpha_m} f\|_2^2 \leq (E+Bm)^m\|f\|_2^2.$$

No normal Bernstein inequality for f. $\exists f \in \operatorname{Ran}\chi_{(-\infty,E]}(H_B)$ with $\partial_1 f \notin L^2(\mathbb{R}^2)$.

Bernstein-type inequality for $|f|^2$:

$$\sum_{\alpha\in\{1,2\}^m}\left\|\partial_{\alpha_1}\partial_{\alpha_2}\dots\partial_{\alpha_m}|f|^2\right\|_1 \leq (E+Bm)^{m/2}\|f\|_2^2.$$

With that, established theory [2] implies that $|f|^2$ locally extends to an analytic function and leads to the spectral inequality for $|f|^2$ in L^1, which is the one for f in L^2.

Comments

Inequalities as (1) are called **Spectral Inequality** or **Unique Continuation Principle**.

Our improvements:

1. We generalize the **non-magnetic** $B = 0$ case. In this case the Theorem is known as **Logvinenko-Sereda-Kovrijkine theorem** [5, 3].
2. Our estimates are **explicit** (and optimal) **in** E.
3. The **relation between** E, B, **and** ℓ is optimal.
4. The **geometric assumption** (thickness) is optimal.

Previous work required S either periodic or an arrangement of balls [1, 6].

$z \mapsto \Phi(z) = \sum_{k=0}^{\infty}\sum_{|\alpha|=k}\frac{\partial^\alpha|f|^2(x_0)}{k!}(z-x_0)$ converges

Application: Controllability

Controlled heat equation with magnetic evolution

$$\begin{cases}\dot u + H_B u = \mathbf{1}_S f, & \text{in } [0,T]\times\mathbb{R}^2,\\ u(0) = u_0 \in L^2(\mathbb{R}^2).\end{cases}$$

Null-controllable in time $T > 0$, if for all **initial states** u_0 there is a **control** $f \in L^2([0,T]\times S)$ such that $u(T) = 0$.

Using the Lebeau-Robbiano method for controllability [**?**], the spectral inequality implies that thickness of S is **sufficient** for controllability. It is essential, that the constant in (1) goes as $\exp(-C\sqrt{E})$. First result on controllability of the Landau-heat equation.

Thickness is also **necessary**. We have identified the **optimal geometric criterion**.

References

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$u(t) = e^{-tH_B}u_0 + \int_0^t e^{-(t-s)H_B}\mathbf{1}f(s)\,ds$

$[\tilde\partial_1, \tilde\partial_2] = iB$

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